Transmission Lines and Waveguides

Unit-1
Transmission Lines-I

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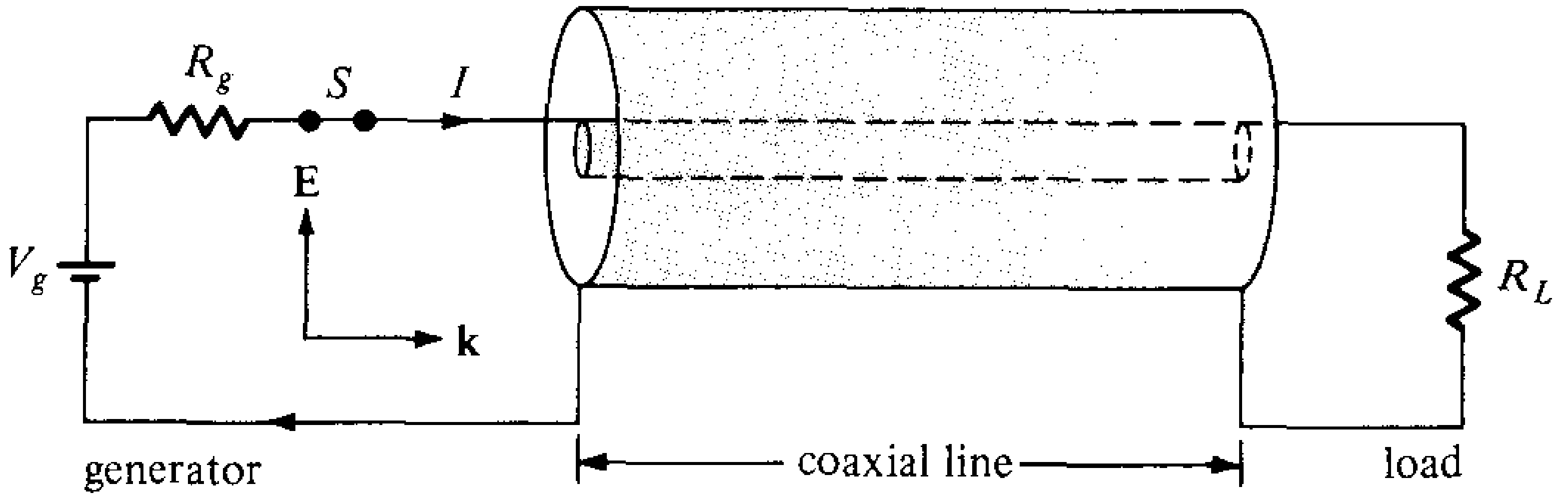
1.Types of transmission lines
2.Transmission line parameters
3.Transmission line equations
4.Secondary constants
5.Characteristics of wave propagation on different transmission lines
   (i)Lossless line
   (ii)Distortion less line
   (iii)Low loss lines
6.Loading

What is Transmission line??



1.Transmission line is defined as path of carrying information in the form of Electro Magnetic Wave(EM wave) from source(generator) to destination(load)

2.Guided structures serve to guide(or direct) the propagation of energy from source to load.

What is Transmission line??

Fig.(a)

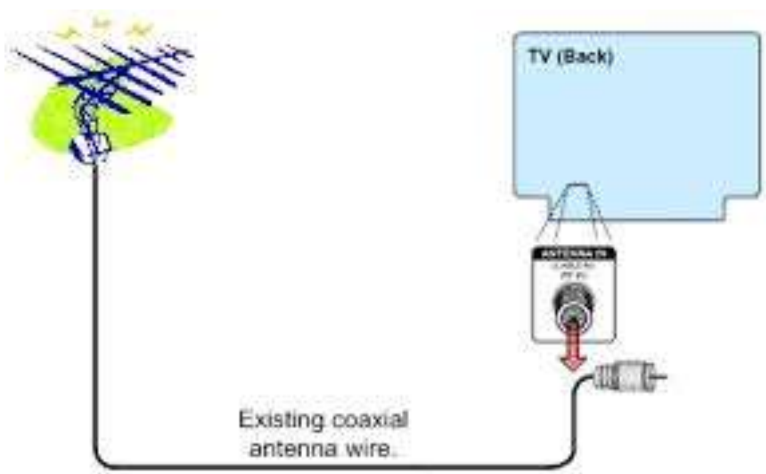


Fig.(b)

Types of transmission lines

(i)parallel wire transmission line

(ii)Coaxial line

(iii)Waveguides

(iv)Optical fiber

(v)Microstrip lines

Types of transmission lines

Parallel wire transmission line

1.These lines are the parallel conductors

2.The conductors are separated by air as the dielectric and mounted on posts or towers

3.Parallel wire transmission line are two types

(i)Low frequency high power line

Ex:Electrical power line

(ii)High frequency Low power lines

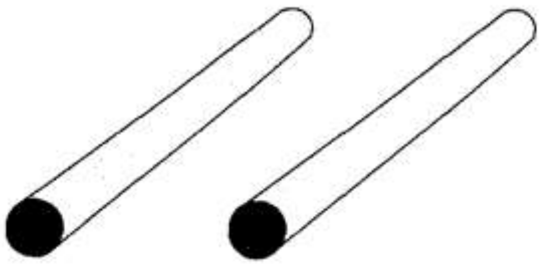
Ex:Telephone lines

4.These lines are balanced with respect to earth and effected by atmospheric conditions like wind, air, ice etc And also possibility of shorting due to flying of objects and birds

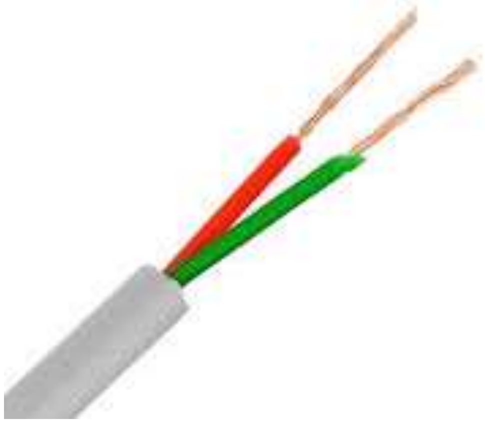
5.These line are not suitable for frequencies above 100MHz

Types of transmission lines

Parallel wire transmission line



(a)



(b)

Fig: Two wire transmission line

Types of transmission lines

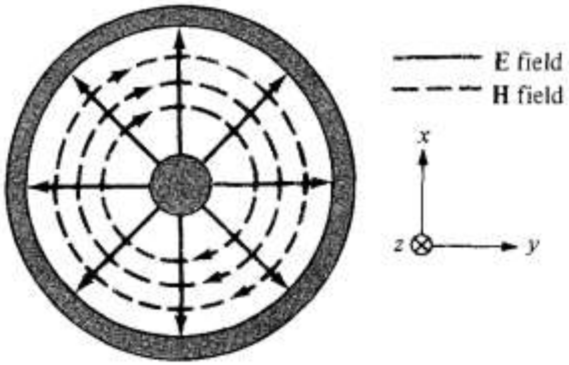
Coaxial transmission line

1.In this type one conductor is hallow tube the second conductor being located inside and coaxial with the tube

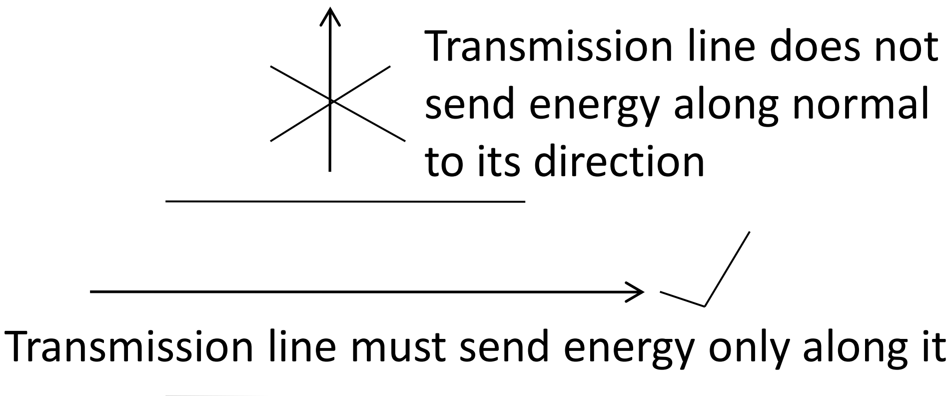
2.The dielectric between the two conductors may be solid or gaseous

3.To avoid radiation losses taken place in open wire line at frequencies beyond 100MHz, a closed Field configuration is employed in coaxial cable by surrounding the inner conductor with an outer cylindrical hallow conductor.

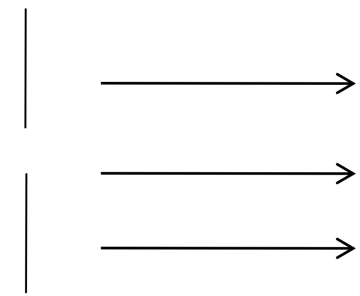
4.Advantage of coaxial cable is that electric and magnetic fields remain confined with in the outer conductor and cannot leak into freespace



Difference between transmission line and antenna



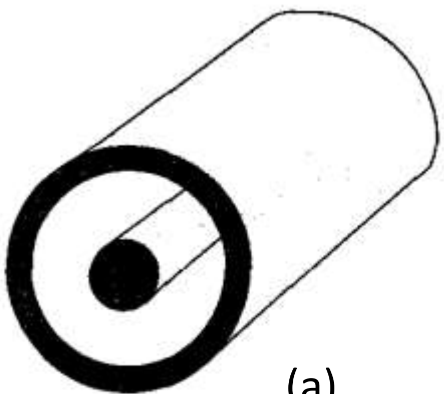
Fig(a) Transmission line



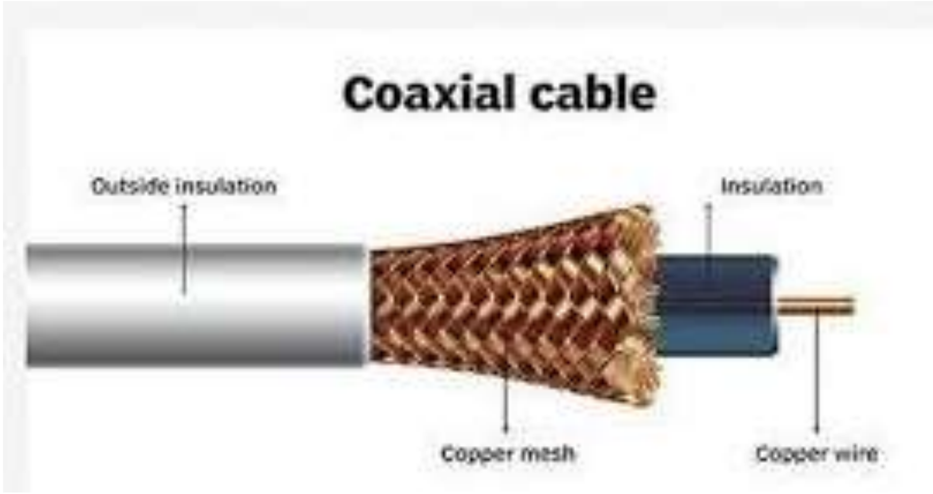
Fig(b) Antenna (Lossy Transmission line)

Types of transmission lines

Coaxial transmission line



(a)



(b)

Fig: Coaxial transmission line

Types of transmission lines

Coaxial transmission line

For lossy dielectrics

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2}-1\right]}$$

As frequency increase, signal attenuation increases

For free space

$$\alpha = 0$$

Types of transmission lines

Waveguides

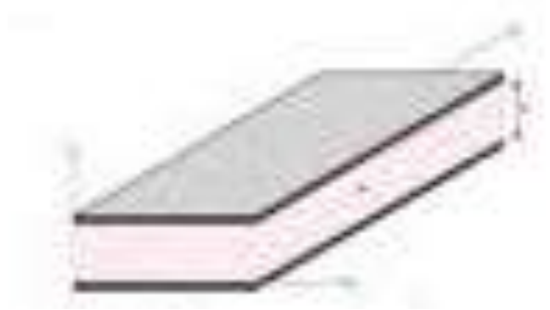
1.A transmission line consisting of a suitable shaped hallow conductor, which may be filled with a dielectric material and is used to guide EM wave of UHF propagated along its length is called a waveguide

2.Waveguide allow to pass different signals simultaneously. However different signals being propagated through a line must have different frequency, but in a waveguide they can have same frequency provided that each is propagated at different mode.

3.This kind of multiplex transmission power handling capacity of waveguide is higher than coaxial cable

Types of transmission lines

Waveguides



Parallel plane waveguide

Rectangular waveguide

Circular waveguide

Skin depth-application

Skin depth is a measure of the depth to which an EM wave can penetrate the medium

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

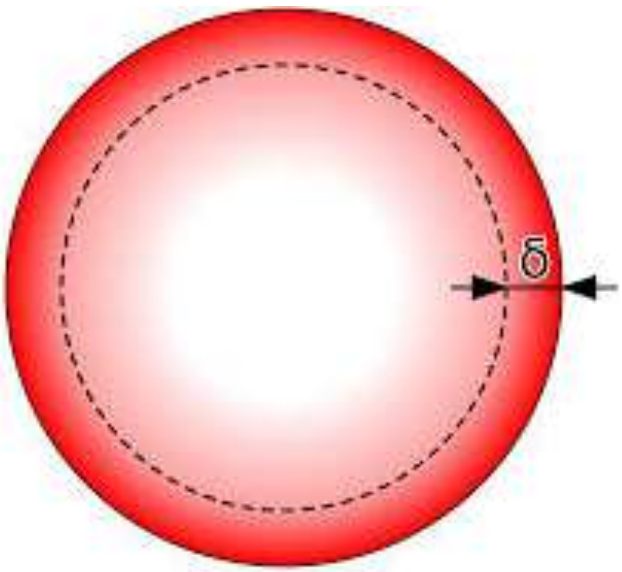
Skin Depth in Copper*

Frequency (Hz)	**10**	**60**	**100**	**500**	$\mathbf{10^4}$	$\mathbf{10^8}$	$\mathbf{10^{10}}$
Skin depth (mm)	20.8	8.6	6.6	2.99	0.66	6.6×10^{-3}	6.6×10^{-4}

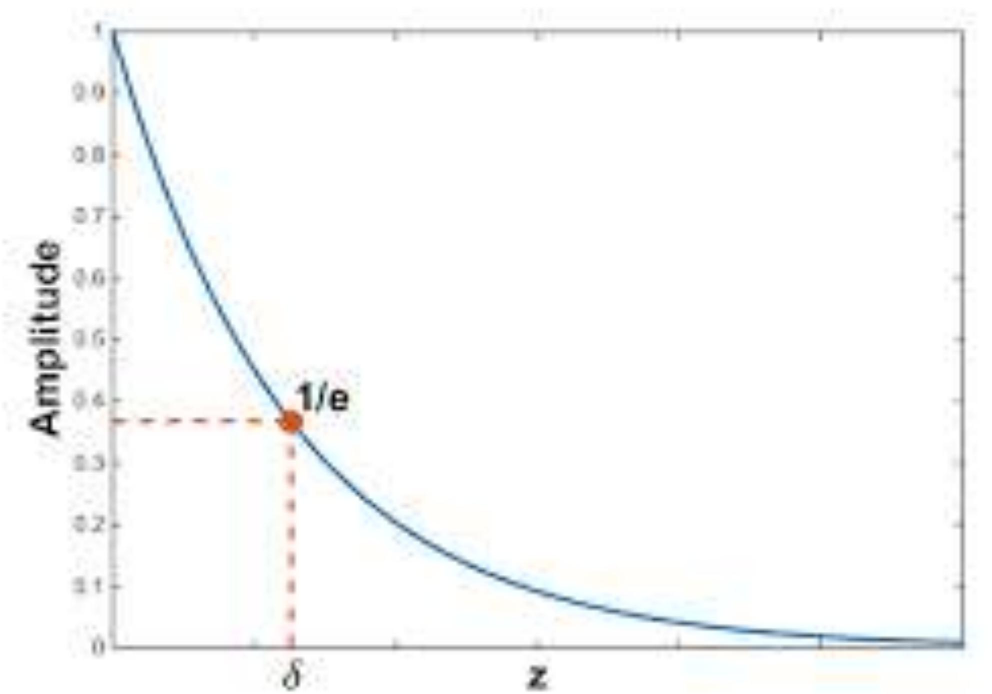
*For copper, $\sigma = 5.8 \times 10^7$ mhos/m, $\mu = \mu_o$, $\delta = 66.1/\sqrt{f}$ (in mm).

➢The skin depth in silver is small, the difference in performance between a pure silver component and a silver plated brass component is negligible, so silver plating is often is used to reduce material cost of waveguide components

Skin depth-application



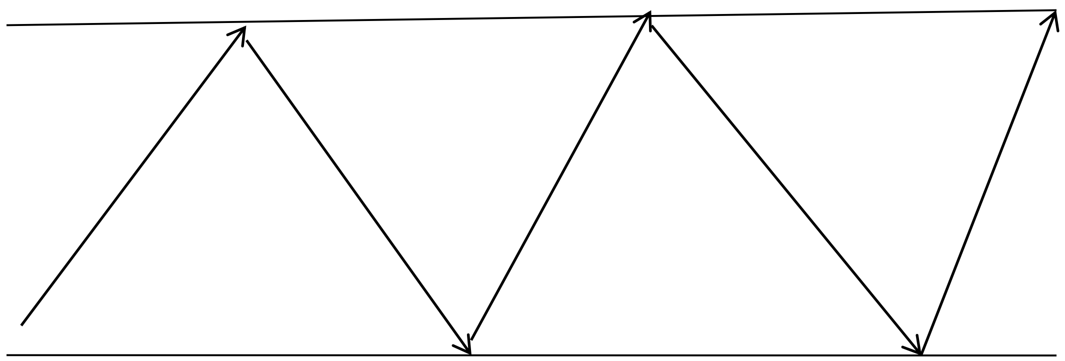
Skin effect



Skin depth

Types of transmission lines

Waveguides



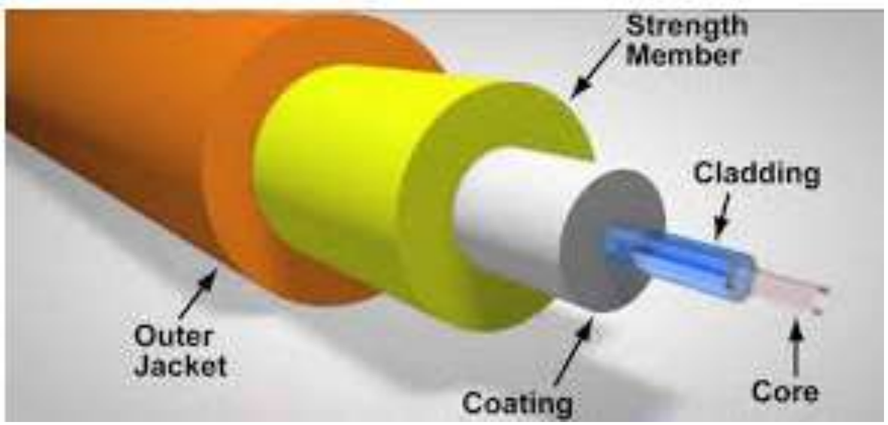
Signal propagation between two conducting walls

Types of transmission lines

Optical fiber

Optical fibers are working on the principle of total internal reflection(TIR)

Optical fiber lines offer several advantages over wire lines
(i)Superior transmission quality
(ii)Higher information carrying capacity due to tremendous bandwidth
(iii)Light weight and smaller size
(iv)Reduced cost and higher security



Core refractive index
Is higher than cladding
refractive index to get
TIR

Optical fiber cable

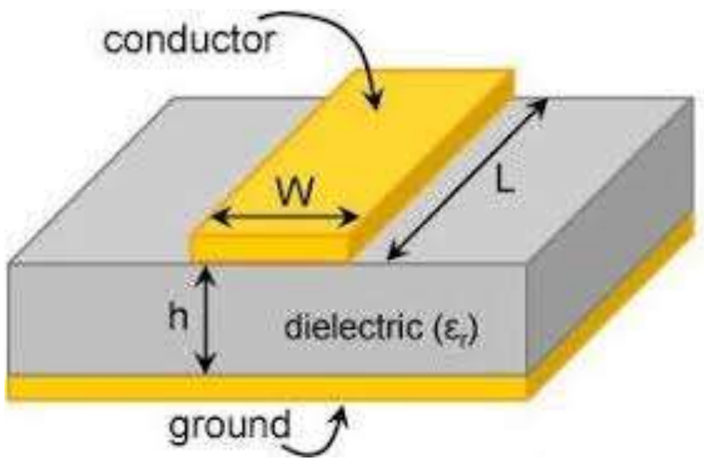
Types of transmission lines

Microstrip lines

1.These lines are particularly important in integrated circuits where metallic strips connecting electronic elements are deposited on dielectric substrates

2.Microstrips are used for circuit components such as filters, couplers, resonators, antennas.

3.These line having greater flexibility



Microstrip line

Types of transmission lines

Comparison of Common Transmission Lines and Waveguides

Characteristic	Coax	Waveguide	Microstrip
Modes: Preferred	TEM	TE_{10}	Quasi-TEM
Other	TM,TE	TM,TE	Hybrid TM,TE
Dispersion	None	Medium	Low
Loss	Medium	Low	High
Power capacity	Medium	High	Low
Physical size	Large	Large	Small
Ease of fabrication	Medium	Medium	Easy
Integration with other components	Hard	Hard	Easy

Transmission line parameters

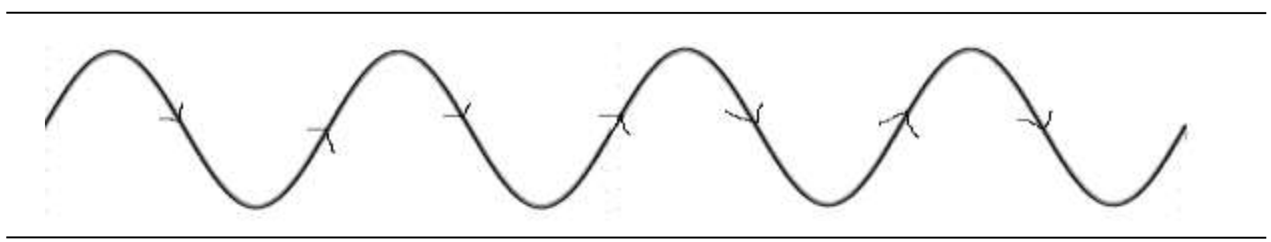


Fig: Two wire transmission line

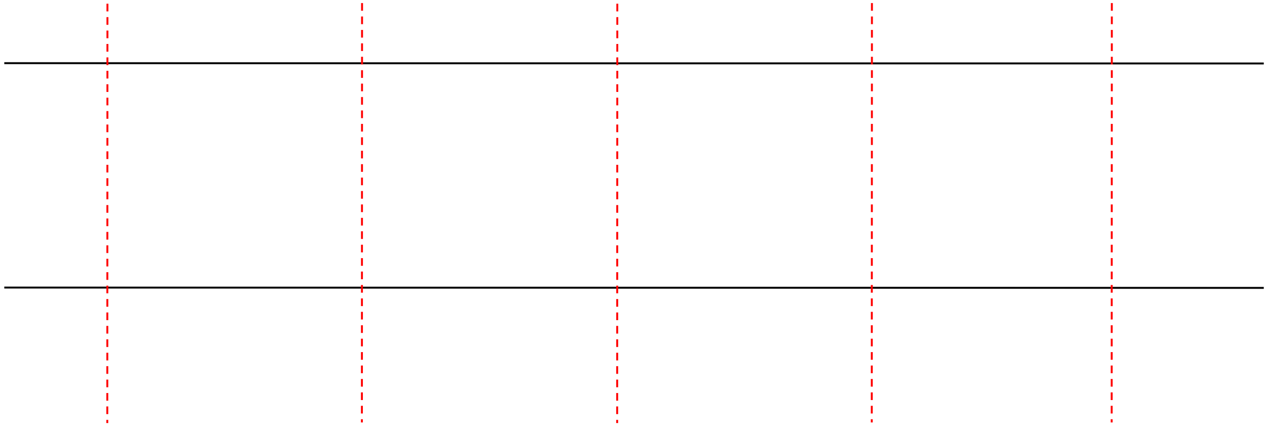
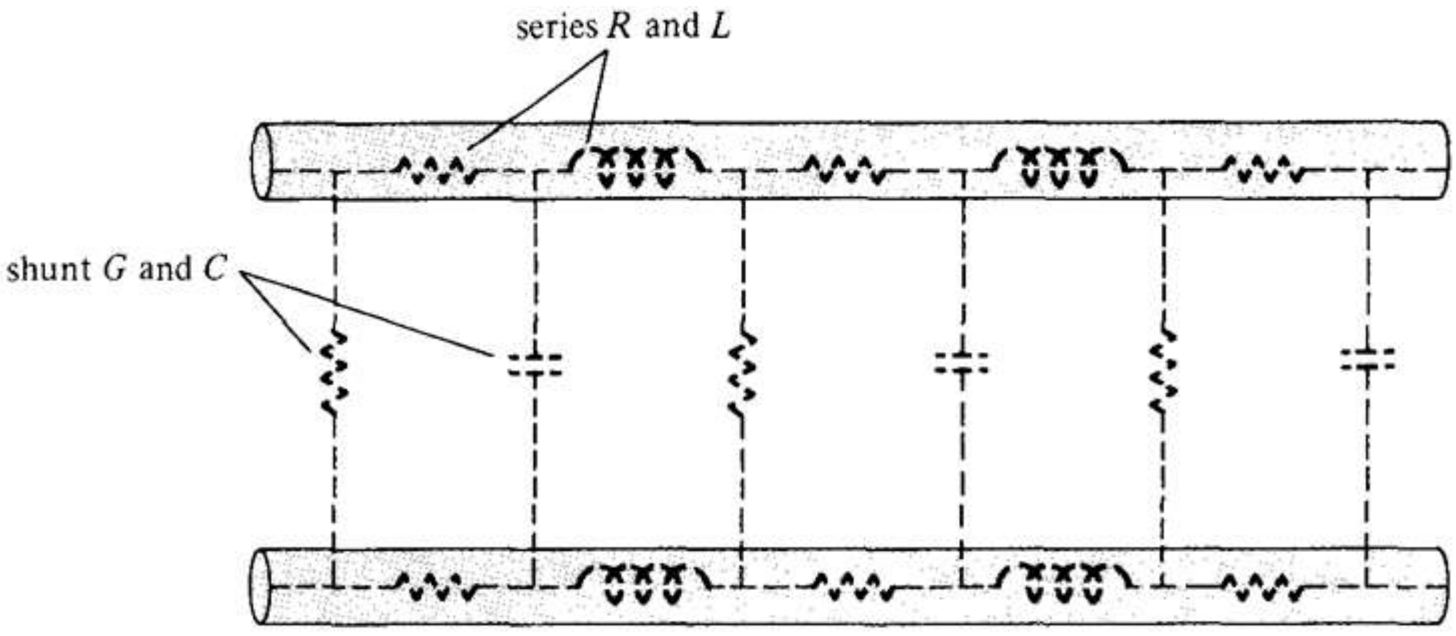


Fig: Dividing of Two wire transmission line into different parts

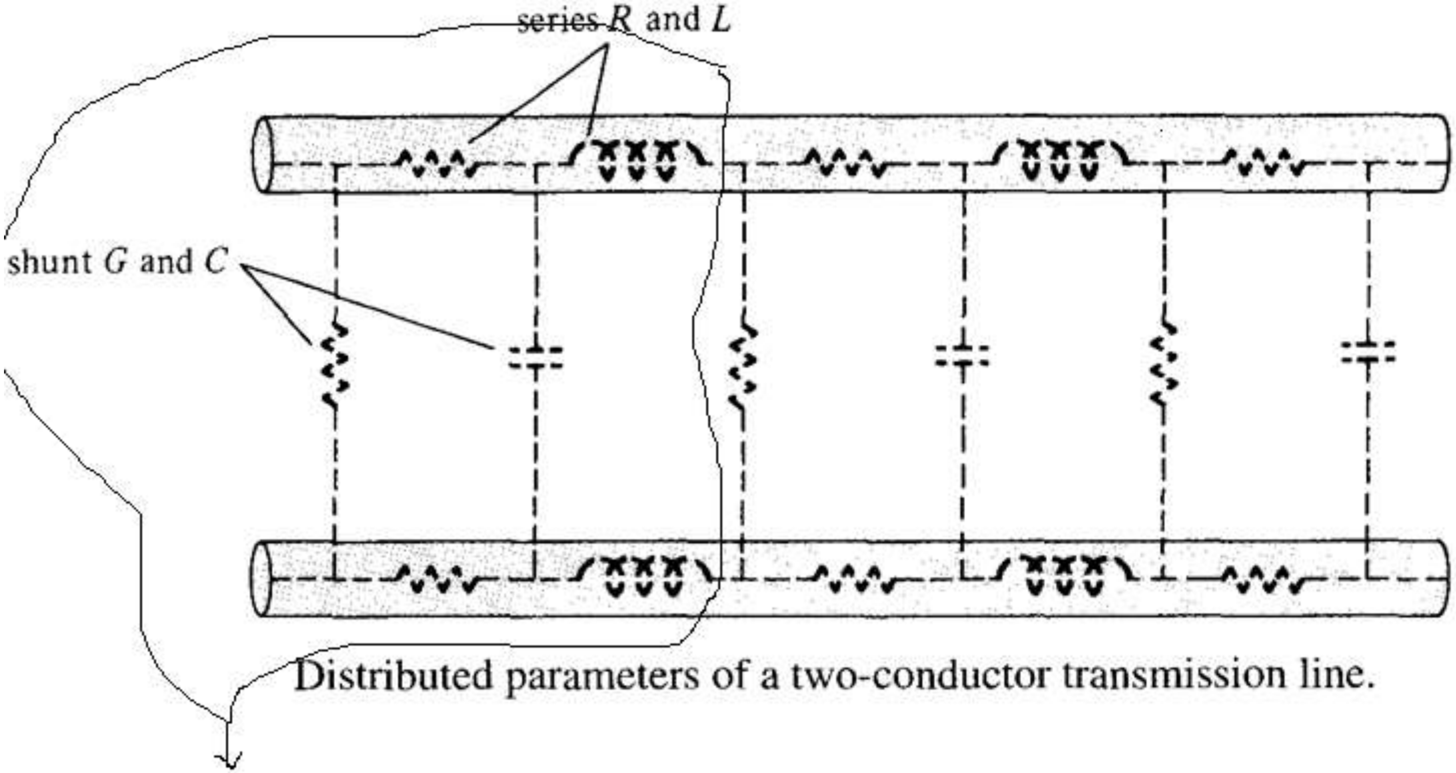
Transmission line parameters



Distributed parameters of a two-conductor transmission line.

R-Resistance of each conductor per unit length (Ω/m)
L-Inductance of each conductor per unit length (H/m)
G-Leakage conductance between two conductors per unit length (mho/m)
C-Capacitance between two conductors per unit length (F/m)

Transmission line parameters



Distributed parameters of a two-conductor transmission line.

1 unit

Transmission line parameters

For the analysis and the design of the transmission lines, it is necessary to have the knowledge of the electric circuit parameters, associated with the transmission lines. The various electric parameters associated with the transmission lines are,

1. Resistance : Depending upon the cross sectional area of the conductors, the transmission lines have the resistance associated with them. The resistance is uniformly distributed all along the length of the transmission line. Its total value depends on the overall length of the transmission line. Hence its value is given per unit length of the transmission line. It is denoted as R and given in ohms per unit length.

$$R = \frac{\rho l}{a}$$

2. Inductance : When the conductors carry the current, the magnetic flux is produced around the conductors. It depends on the magnitude of the current flowing through the conductors. The flux linkages per ampere of current, gives rise to the effect called inductance of the transmission line. It is also distributed all along the length of the transmission line. It is denoted as L and measured in henry per unit length of the transmission line.

$$\varphi \alpha I \quad ; \quad \varphi = LI \quad ; \quad L = \frac{\varphi}{I}$$

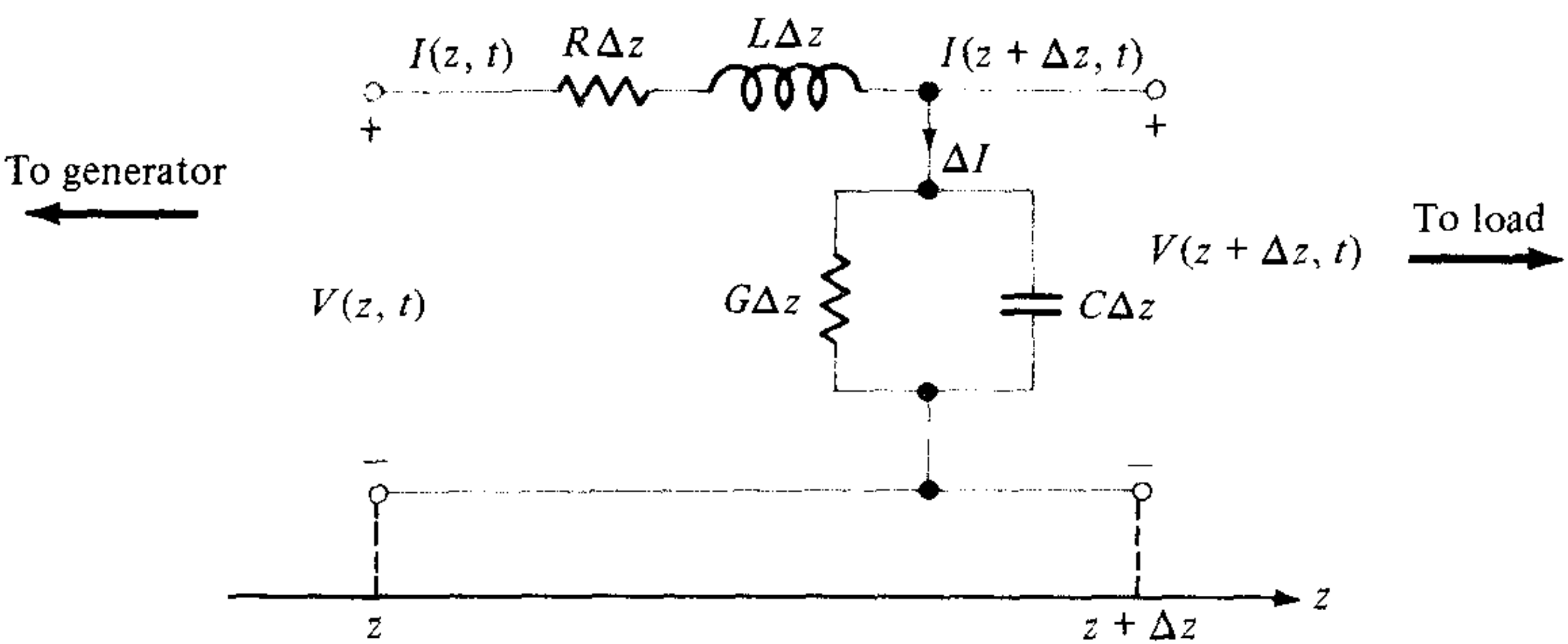
Transmission line parameters

3. Conductance : The dielectric in between the conductors is not perfect. Hence a very small amount of current flows through the dielectric called displacement current. This is nothing but a leakage current and this gives rise to a leakage conductance associated with the transmission line. It exists between the conductors and distributed along the length of the transmission line. It is denoted as G and measured in mho per unit length of the transmission line.

4. Capacitance : The transmission line consists of two parallel conductors, separated by a dielectric like air. Such parallel conductors separated by an insulating dielectric produces a capacitive effect. Due to this, there exists a capacitance associated with the transmission line which is also distributed along the length of the conductor. It is denoted as C and measured in farads per unit length of the transmission line.

$$C = \varepsilon \frac{A}{d}$$

Transmission line equations



L-type equivalent circuit model of a differential length Δz of a two-conductor transmission line.

Transmission line equations

By applying KVL to the outer loop of circuit

$$V(z,t) = R\,\Delta z\,I(z,t) + L\,\Delta z\,\frac{\partial I(z,t)}{\partial t} + V(z+\Delta z,t)$$

$$-\frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = R\,I(z,t) + L\,\frac{\partial I(z,t)}{\partial t}$$

If $\Delta z \rightarrow 0$ $\quad \frac{V(z+\Delta z,t)-V(z,t)}{\Delta z} = \frac{\partial V(z,t)}{\partial z}$

$$-\frac{\partial V(z,t)}{\partial z} = RI(z,t) + L\,\frac{\partial I(z,t)}{\partial t} \qquad \text{——— (1)}$$

Transmission line equations

By applying KCL to the main node of circuit

I(z, t) = I(z + Az, t) + ΔI

$$-\frac{I(z+\Delta z,t)-I(z,t)}{\Delta z} = G\,V(z+\Delta z,t) + C\frac{\partial V(z+\Delta z,t)}{\partial t}$$

If $\Delta z \to 0 \quad \frac{I(z+\Delta z,t)-I(z,t)}{\Delta z} = \frac{\partial I(z,t)}{\partial z}$

$$-\frac{\partial I(z,t)}{\partial z} = G\,V(z,t) + C\frac{\partial V(z,t)}{\partial t} \qquad \text{———} \quad (2)$$

Transmission line equations

If we assume time harmonic dependence

$$V(z,t) = V(z)\mathrm{e}^{j\omega t} \quad ; \quad \mathrm{I}(z,t) = I(z)\mathrm{e}^{j\omega t}$$

Substitute above two equations in (1), (2) respectively

$$-\frac{dV}{dz} = (R + j\omega L)\,\mathrm{I} \qquad \text{———— (3)}$$

$$-\frac{dI}{dz} = (\mathrm{G} + j\omega C)\,\mathrm{V} \qquad \text{———— (4)}$$

Now differentiate equation (3) w.r.t z

$$-\frac{d}{dz}\left(\frac{dV}{dz}\right) = (R + j\omega L)\frac{dI}{dz}$$

Transmission line equations

$$-\frac{d^2V}{dz^2} = -(R + j\omega L)(G + j\omega C)V$$

Let $(R + j\omega L)(G + j\omega C) = \gamma^2$ ——— (5)

$$\frac{d^2V}{dz^2} - \gamma^2 V = 0 \qquad \text{———} \quad (6)$$

Now differentiate equation (4) w.r.t z

$$-\frac{d}{dz}\left(\frac{dI}{dz}\right) = (G + j\omega C)\frac{dV}{dz}$$

Transmission line equations

$$-\frac{d^2I}{dz^2} = -(R + j\omega L)(G + j\omega C)\,I$$

$$\frac{d^2I}{dz^2} - \gamma^2 I = 0 \qquad \text{———} \qquad (7)$$

Final Voltage and Current equations of transmission line are

$$\left\{\begin{aligned} &\frac{d^2V}{dz^2} - \gamma^2 V = 0 \qquad \text{———} \qquad (6) \\ &\frac{d^2I}{dz^2} - \gamma^2 I = 0 \qquad \text{———} \qquad (7) \end{aligned}\right\}$$

Where $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$

Secondary Constants

Propagation constant

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\gamma = \alpha + j\beta$$

Where γ is propagation constant which tells about propagation characteristics of EM wave

Where α is attenuation constant which is measure of attenuation of EM wave

Where β is phase constant which is measure of phase variation per unit wave length

γ Units- 1/m

α Units- dB/m or NP/m

β Units- rad/m

1 Neper =8.686dB

$\beta = \frac{2\pi}{\lambda}$; λ is wave length

$v = \frac{\omega}{\beta}$; v Is velocity of wave propagatior

ω is angular frequency

ω=2πf ; f is frequency

Secondary Constants

Propagation constant

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$$

$$|\gamma| = \sqrt{(|R + j\omega L|)(|G + j\omega C|)} = |\alpha + j\beta|$$

$$\sqrt{\alpha^2 + \beta^2} = \sqrt[4]{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}$$

$$\alpha^2 + \beta^2 = \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} \quad \text{——— (i)}$$

$$(\alpha + j\beta)^2 = (R + j\omega L)(G + j\omega C)$$

$$\alpha^2 + 2j\alpha\beta + j^2\beta^2 = RG + j\omega LG + j\omega RC + j^2\omega^2 LC$$

Secondary Constants

Propagation constant

$$\alpha^2 - \beta^2 + j2\alpha\beta = \left(RG - \omega^2 LC\right) + j\omega(LG + RC)$$

Equating real parts,

$$\alpha^2 - \beta^2 = \left(RG - \omega^2 LC\right) \qquad \text{———— (ii)}$$

Adding (i) and (ii)

$$2\alpha^2 = \sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)} + \left(RG - \omega^2 LC\right)$$

$$\alpha = \sqrt{\frac{1}{2}\left\{\sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)} + \left(RG - \omega^2 LC\right)\right\}}$$

Secondary Constants

Propagation constant

Subtracting (ii) from (i)

$$2\beta^2 = \sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)} - (RG - \omega^2 LC)$$

$$\beta = \sqrt{\frac{1}{2}\left\{\sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)} - (RG - \omega^2 LC)\right\}}$$

Secondary Constants

Characteristic impedance

The solutions of the linear homogeneous differential equations (6), (7) are

$$V(z) = \underset{\longrightarrow +z}{V_0^+ e^{-\gamma z}} + \underset{-z \longleftarrow}{V_0^- e^{\gamma z}} \qquad \text{——— (8)}$$

$$I(z) = \underset{\longrightarrow +z}{I_0^+ e^{-\gamma z}} + \underset{-z \longleftarrow}{I_0^- e^{\gamma z}} \qquad \text{——— (9)}$$

Where $V_0^+, V_0^-, I_0^+, I_0^-$ are wave amplitudes; the + and – signs, respectively, denote wave traveling along +z and –z directions, as is also indicated by arrows

Secondary Constants

Characteristic impedance

The characteristic impedance Z_0 of the line is the ratio of positively traveling voltage wave to current wave at any point on the line

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-}$$

Now substitute equation (8) , (9) into equation (3)

$$-\frac{d}{dz}(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}) = (R + j\omega L)(I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z})$$

$$V_0^+ \gamma e^{-\gamma z} - V_0^- \gamma e^{\gamma z} = (R + j\omega L) I_0^+ e^{-\gamma z} + (R + j\omega L) I_0^- e^{\gamma z}$$

Secondary Constants

Characteristic impedance

Compare $e^{\gamma z}$ coefficients

$$V_0^+ \gamma e^{-\gamma z} - V_0^- \gamma e^{\gamma z} = (R + j\omega L) I_0^+ e^{-\gamma z} + (R + j\omega L) I_0^- e^{\gamma z}$$

Compare $e^{-\gamma z}$ coefficients

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma}$$

Secondary Constants

Characteristic impedance

$$Z_0 = \frac{R + j\omega L}{\sqrt{(R + j\omega L)(G + j\omega C)}}$$

$$\left\{ Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = R_0 + jX_0 \right\}$$

Where R_0 and X_0 Real and imaginary parts of Z_0

The reciprocal of Z_0 Is the characteristic admittance $Y_0 = \frac{1}{Z_0}$

Characteristics of wave propagation on different transmission lines

The transmission line considered thus far in this section is the *lossy* type in that the conductors comprising the line are imperfect ($\sigma_c \neq \infty$) and the dielectric in which the conductors are embedded is lossy ($\sigma \neq 0$). Having considered this general case, we may now consider three special cases of lossless transmission line, distortionless line, Low loss line

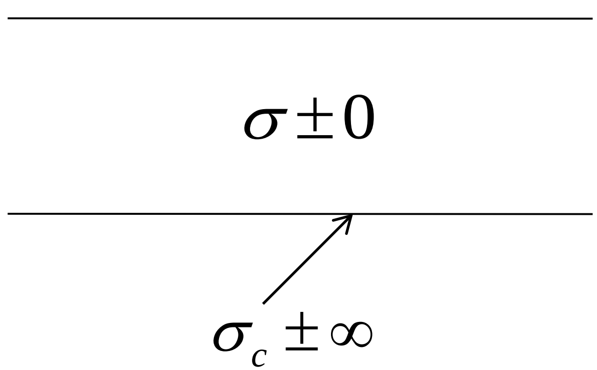


Fig: Lossy transmission line

Characteristics of wave propagation on different transmission lines

Lossless Line (R = G = 0)

A **transmission line** is said to be **lossless** if the conductors of the line are perfect ($\sigma_c \simeq \infty$) and the dielectric medium separating them is lossless ($\sigma \simeq 0$).

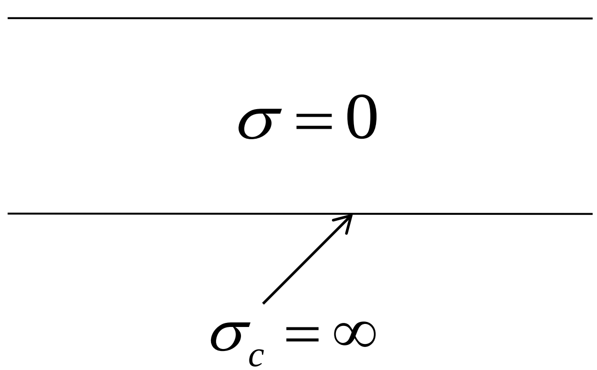


Fig: Lossless transmission line

Characteristics of wave propagation on different transmission lines

Lossless Line (R = G = 0)

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

If R=G=0

$$\gamma = j\omega\sqrt{LC} = \alpha + j\beta$$

Compare real and imaginary parts

$$\alpha = 0 \quad ; \quad \beta = \omega\sqrt{LC}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = R_0 + jX_0$$

$$Z_0 = R_0 = \sqrt{\frac{L}{C}} \quad ; \quad X_0 = 0$$

$$v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}}$$

$$v = \frac{1}{\sqrt{LC}} = f\lambda$$

Characteristics of wave propagation on different transmission lines

Lossless Line (R = G = 0)

Conclusions

1. Propagation constant is purely imaginary
2. Signal attenuation is equal to zero
3. Phase constant is linearly dependant on frequency
4. Velocity of wave propagation is constant
5. Characteristic impedance is purely real

Characteristics of wave propagation on different transmission lines

Distortion less line(R/L = G/C)

A **distortionless line** is one in which the attenuation constant α is frequency independent while the phase constant β is linearly dependent on frequency.

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\gamma = \sqrt{RG\left(1 + \frac{j\omega L}{R}\right)\left(1 + \frac{j\omega C}{G}\right)}$$

$$= \sqrt{RG}\left(1 + \frac{j\omega C}{G}\right) = \alpha + j\beta$$

$$= \sqrt{RG} + j\omega \frac{C}{G}\sqrt{RG}$$

Characteristics of wave propagation on different transmission lines

Distortion less line(R/L = G/C)

$$\gamma = \sqrt{RG} + j\omega C\sqrt{\frac{R}{G}}$$

$$\gamma = \sqrt{RG} + j\omega C\sqrt{\frac{L}{C}}$$

$$\gamma = \sqrt{RG} + j\omega\sqrt{LC} = \alpha + j\beta$$

$$\alpha = \sqrt{RG}$$

$$\beta = \omega\sqrt{LC}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = R_0 + jX_0$$

$$Z_0 = \sqrt{\frac{R(1 + j\omega L/R)}{G(1 + j\omega C/G)}} = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}} = R_0 + jX_0$$

$$R_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}, \qquad X_0 = 0$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = f\lambda$$

Characteristics of wave propagation on different transmission lines

Distortion less line(R/L = G/C)

Conclusions

1.Attenuation does not depend on frequency and it is constant
2.Phase constant is linearly dependant on frequency
3.The phase velocity is independent of frequency
4.Velocity and characteristic impedance remain same as for lossless lines
5.A lossless is also a distortion less line, but a distortion less line is not necessarily lossless.
6.Lossless lines are desirable in power transmission, telephone lines are required to be distortion less

Characteristics of wave propagation on different transmission lines

Low loss line(R<<ωL, G<<ωC)-UHF lines

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\gamma = j\omega\sqrt{LC}(1 + \frac{R}{j\omega L})^{\frac{1}{2}}(1 + \frac{G}{j\omega C})^{\frac{1}{2}}$$

$$\approx j\omega\sqrt{LC}(1 + \frac{R}{2j\omega L})(1 + \frac{G}{2j\omega C})$$

$$\approx j\omega\sqrt{LC}[1 + \frac{1}{2j\omega}(\frac{R}{L} + \frac{G}{C})]$$

$$\approx \frac{1}{2}(\frac{R}{L}\sqrt{LC} + \frac{G}{C}\sqrt{LC}) + j\omega\sqrt{LC} = \alpha + j\beta$$

$$\alpha \approx \frac{1}{2}(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}})$$

$$\beta \approx \omega\sqrt{LC}$$

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

Characteristics of wave propagation on different transmission lines

Low loss line(R<<ωL, G<<ωC)-UHF lines

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = R_0 + jX_0$$

$$Z_0 = \sqrt{\frac{L}{C}}(1+\frac{R}{j\omega L})^{1/2}(1+\frac{G}{j\omega C})^{-1/2}$$

$$\approx \sqrt{\frac{L}{C}}(1+\frac{R}{2j\omega L})(1-\frac{G}{2j\omega C})$$

$$\approx \sqrt{\frac{L}{C}}[1+\frac{1}{2j\omega}(\frac{R}{L}-\frac{G}{C})] = \mathrm{R}_0 + jX_0$$

$$R_0 \approx \sqrt{\frac{L}{C}}$$

$$X_0 \approx 0$$

Problems

1. An air line has characteristic impedance of 70 Ω and phase constant of 3 rad/m at 100 MHz. Calculate the inductance per meter and the capacitance per meter of the line.

Sol: An air line can be regarded as a lossless line since $\sigma \simeq 0$. Hence

$$R = 0 = G \qquad \text{and} \qquad \alpha = 0$$

$$Z_o = R_o = \sqrt{\frac{L}{C}} \qquad\qquad \beta = \omega\sqrt{LC}$$

$$\frac{R_o}{\beta} = \frac{1}{\omega C}$$

$$C = \frac{\beta}{\omega R_o} = \frac{3}{2\pi \times 100 \times 10^6 (70)} = 68.2 \text{ pF/m}$$

$$L = R_o^2 C = (70)^2 (68.2 \times 10^{-12}) = 334.2 \text{ nH/m}$$

Problems

2. A distortionless line has $Z_o = 60\ \Omega$, $\alpha = 20$ mNp/m, $u = 0.6c$, where c is the speed of light in a vacuum. Find R, L, G, C, and λ at 100 MHz.

Sol:

$$RC = GL \qquad \text{or} \qquad G = \frac{RC}{L}$$

$$Z_o = \sqrt{\frac{L}{C}}$$

$$\alpha = \sqrt{RG} = R\sqrt{\frac{C}{L}} = \frac{R}{Z_o} \longrightarrow R = \alpha Z_o$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

$$R = \alpha Z_o = (20 \times 10^{-3})(60) = 1.2\ \Omega/\text{m}$$

$$L = \frac{Z_o}{u} = \frac{60}{0.6\,(3 \times 10^8)} = 333\ \text{nH/m}$$

$$G = \frac{\alpha^2}{R} = \frac{400 \times 10^{-6}}{1.2} = 333\ \mu\text{S/m}$$

Problems

$$uZ_o = \frac{1}{C}$$

$$C = \frac{1}{uZ_o} = \frac{1}{0.6\,(3 \times 10^8)\,60} = 92.59 \text{ pF/m}$$

$$\lambda = \frac{u}{f} = \frac{0.6\,(3 \times 10^8)}{10^8} = 1.8 \text{ m}$$

Problems

3. A telephone line has $R = 30\ \Omega/\text{km}$, $L = 100\ \text{mH/km}$, $G = 0$, and $C = 20\ \mu\text{F/km}$. At $f = 1\ \text{kHz}$, obtain:

   (a) The characteristic impedance of the line

   (b) The propagation constant

   (c) The phase velocity

4. A transmission line operating at 500 MHz has $Z_o = 80\ \Omega$, $\alpha = 0.04\ \text{Np/m}$, $\beta = 1.5\ \text{rad/m}$. Find the line parameters R, L, G, and C.

Loading

It is seen earlier that if the primary constants of a line, mutually satisfy the relationship RC = LG then the distortionless transmission results.

For a practical line, R/G is always more than L/C and hence the signal is distorted. Thus the preventive remedy is to make the condition $\frac{R}{G} = \frac{L}{C}$ satisfy artificially.

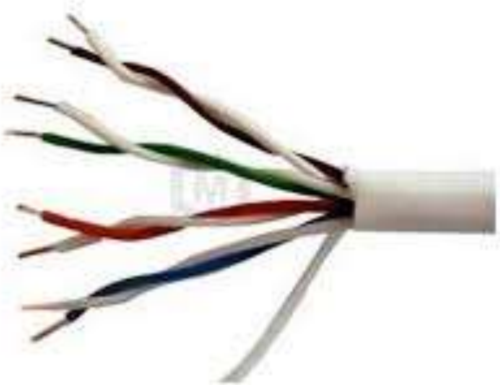
To satisfy the condition, it is necessary to reduce R/G or increase L/C. Let us consider all the possibilities. To reduce R/G, it is necessary to decrease R or increase G. The resistance R can be decreased by increasing the area of cross-section i.e. diameter of the conductors. This increases the size and cost of the line. Hence this possibility is uneconomical.

Loading

To increase G, it is necessary to use poor insulators. To get poor insulator is easy and economical but from the receiving end point of view, increase in G is very much uneconomical. When G is increased, the leakage of the signal will increase, though it becomes distortionless. So quality improves but quantity decreases. Thus increase in G is quality at the cost of quantity. The signal at receiving end must activate the receivers. But if leakage is more, then received signal becomes so weak that amplifiers are required at the intermediate stages. This makes the design complicated. Hence advice of increasing G to reduce ratio R/G is 'penny wise pound foolish' advice. It is the worst advice and hence this possibility is ruled out in practice.

Now to increase L/C, it is necessary to increase L or decrease C. If C is to be reduced, then the seperation between the lines will be more. Thus the brackets which were carrying previously more wires will now carry very less number of wires due to increased separation. Hence more number of brackets are required.

Loading



(a)

(b)

Fig: Telephone lines with (a) 5 pair cables (b) 100 pair cables

Loading

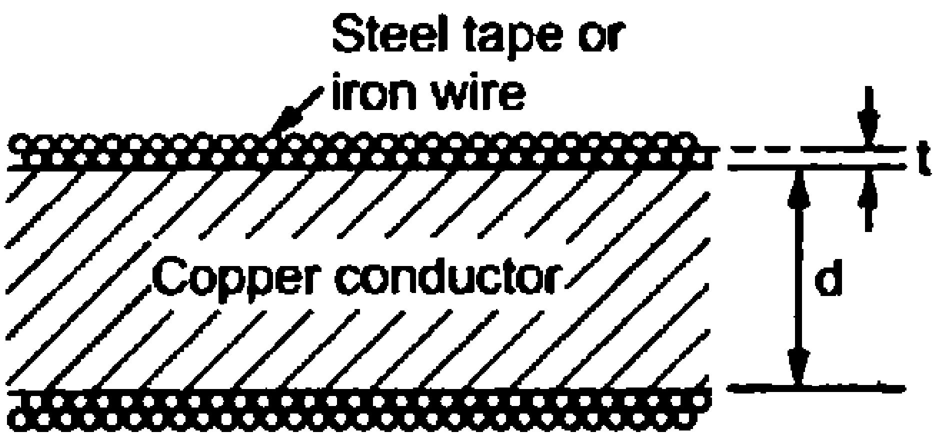
Thus the only alternative left is to increase L. This is opted in practice. The process of increasing the inductance L of a line artificially is called **loading of a line.** And such a line is called **loaded line.**

There are two methods of loading a line
1.Continouos loading
2.Lumped loading

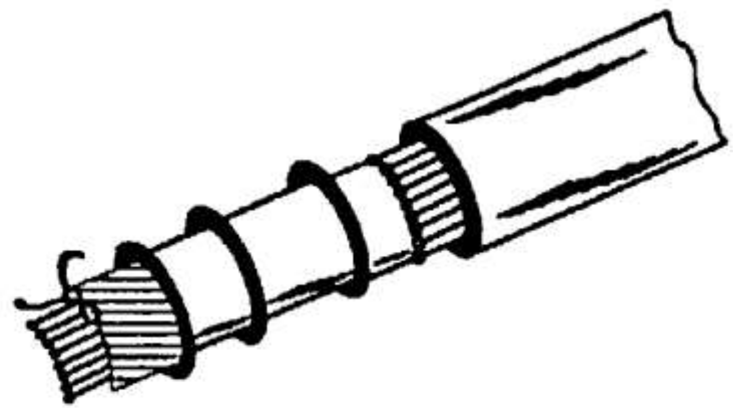
Loading

Continuous loading

In this method of loading, to increase the inductance, on each conductor the tapes of magnetic material having high permeability such as permalloy or μ-metal are wound. This is shown in the Fig (a) while the loaded cable is shown in the Fig. (b).



(a) Continuous loading



(b) Continuously loaded cable

Loading

Advantages of continuous loading

1. The attenuation to the signal is independent of the frequency and it is same to all the frequencies
2. The increase in the inductance up to 100mH per unit length of the line is possible

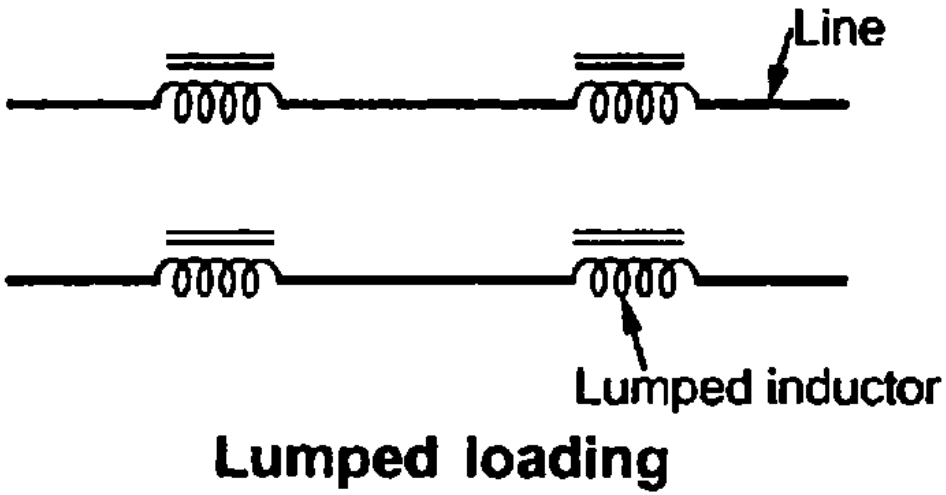
Disadvantages of continuous loading

1. The method is very costly
2. Existing lines cannot be modified by this method. Hence total replacement of the existing cables by the new cables wound with magnetic tapes is required. This again costly and uneconomical.
3. Extreme precision care must be taken while manufacturing continuously loaded cable, otherwise it becomes irregular.
4. All along the conductor, there will be huge mass of iron. Thus for ac signals there will be large eddy current and hysteresis losses. The eddy current losses increase directly with square of frequency while the hysteresis losses increase directly with the frequency. Hence overall this puts the upper limit to increase inductance.

This method is not used for the landlines but are preferred for the submarine cables

Loading

Lumped Loading



In this type of loading , the inductors are introduced in lumps at the uniform distances, in the line. Such inductors are called lumped inductors. The inductors are introduced in both the limps to keep the line as balanced circuit. The lumped inductors are in the form of coils called loaded coils.

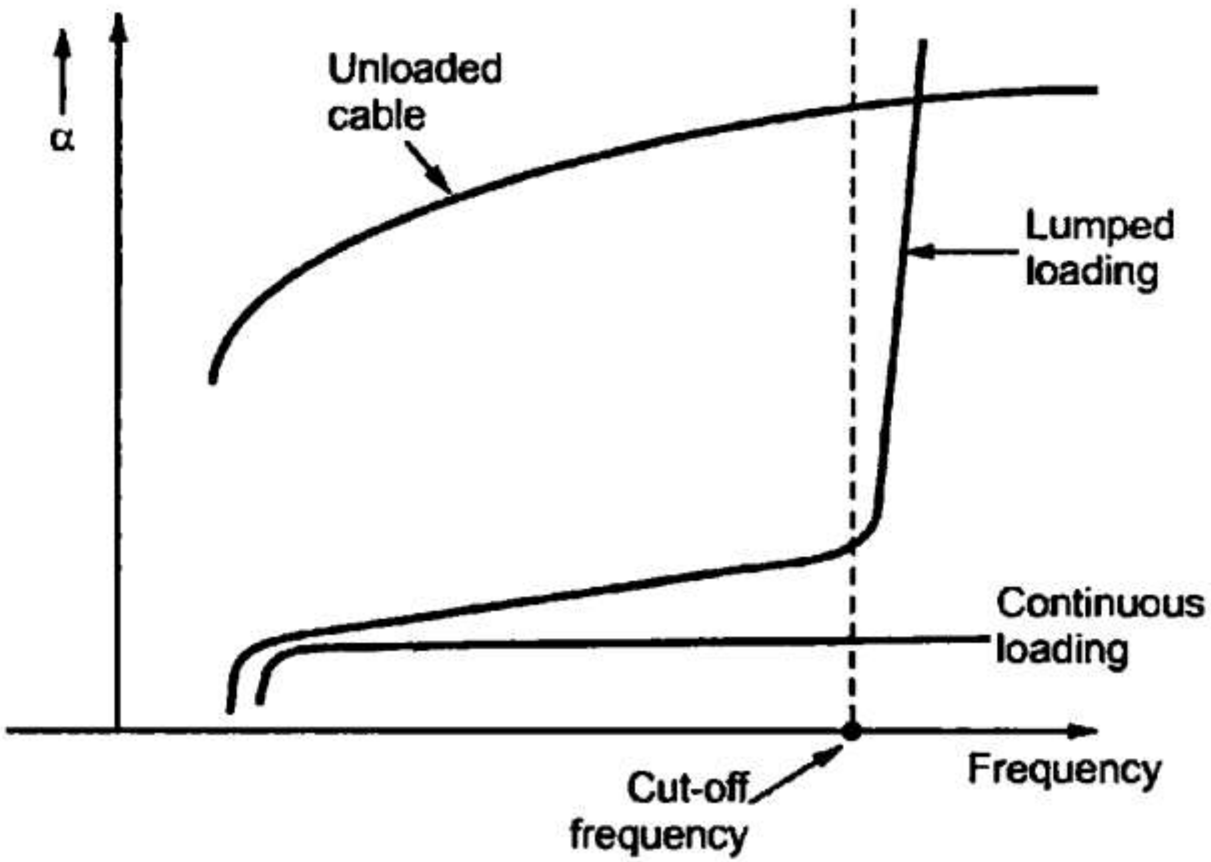
Loading

Lumped Loading

Advantages
1.The cost involved is less
2.With this method the existing lines can be tackled and modified
3.Hysteresis and eddy current losses are small

Disadvantages
1.The attenuation increases considerably after the cutoff frequency

Loading



Attenuation frequency characteristics

Condition for distortion less line

Condition for distortion less line or minimum attenuation

$$\frac{d\alpha}{dL} = 0 \quad \text{or} \quad \frac{d\alpha}{dC} = 0$$

$$\alpha = \sqrt{\frac{1}{2}\left\{\sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)} + \left(RG - \omega^2 LC\right)\right\}}$$

$$\frac{d\alpha}{dL} = \frac{d}{dL}\left\{\frac{1}{2}\left[\left\{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)\right\}^{\frac{1}{2}} + RG - \omega^2 LC\right]\right\}^{\frac{1}{2}}$$

Condition for distortion less line

$$\frac{d\alpha}{dL} = \frac{1}{2}\left\{\frac{1}{2}\left[\left\{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)\right\}^{\frac{1}{2}}+RG-\omega^2LC\right]\right\}^{\frac{1}{2}-1}$$

$$\times\frac{1}{2}\left\{\left[\frac{1}{2}\left\{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)\right\}^{\frac{1}{2}-1}\right]\left[2\omega^2L\left(G^2+\omega^2C^2\right)\right]-\omega^2C\right\}$$

$$= \frac{1}{2}\frac{\frac{1}{2}\left\{\frac{1}{2}\frac{2\omega^2L\left(G^2+\omega^2C^2\right)}{\sqrt{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)}}-\omega^2C\right\}}{\sqrt{\frac{1}{2}\left[\sqrt{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)}+RG-\omega^2LC\right]}} = 0$$

Condition for distortion less line

$$\frac{\omega^2 L(G^2+\omega^2 C^2)}{\sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)}} - \omega^2 C = 0$$

$$\frac{L(G^2+\omega^2 C^2)}{\sqrt{(R^2+\omega^2 L^2)(G^2+\omega^2 C^2)}} = C$$

$$L\sqrt{\frac{G^2+\omega^2 C^2}{R^2+\omega^2 L^2}} = C$$

Condition for distortion less line

$$L\sqrt{(G^2+\omega^2C^2)} = C\sqrt{(R^2+\omega^2L^2)}$$

$$L^2(G^2+\omega^2C^2) = C^2(R^2+\omega^2L^2)$$

$$L^2G^2+\omega^2L^2C^2 = C^2R^2+\omega^2L^2C^2$$

$$L^2 = \frac{C^2R^2}{G^2} \longrightarrow L = \frac{CR}{G}$$

$$\left[\frac{R}{L} = \frac{G}{C}\right]$$

Questions

1. Describe the different types of transmission lines with schematic diagrams
2. Show that attenuation is constant and phase constant is linearly dependant on frequency in a distortion less transmission line
3. Summarize the characteristics of wave propagation in loss less and distortion less transmission lines.
4. Show that phase constant is same in lossless, distortion less and low loss transmission lines.
5. Derive the condition which is used for minimum attenuation in transmission line.
6. Describe the equivalent circuit of two wire transmission line
7. Differentiate the lossless, distortion less, low loss transmission lines.
8. Determine the line parameters R,L,G,C for a distortion less line with $\gamma=0.04+j15$ /m, $Z_0=80\ \Omega$, f=500MHz.
9. Discuss primary constants of transmission line with the use of equivalent model
10. Derive the characteristic impedance of transmission line in terms of its line constants.

END OF UNIT-1

Transmission Lines and Waveguides

Unit-2
Transmission Lines-II

Contents

1.Input impedance of transmission line
2.(i)Reflection coefficient
(ii)VSWR
3.(i)SC lines
(ii)OC lines
(Iii)Matched line
4.Infinite line
5.λ/8, λ/4, λ/2 lines
6.Power in a transmission line
7.Smith chart
8.Matching Techniques
(i)Quarter wave transformer
(ii)Single stub matching

Input impedance of transmission line

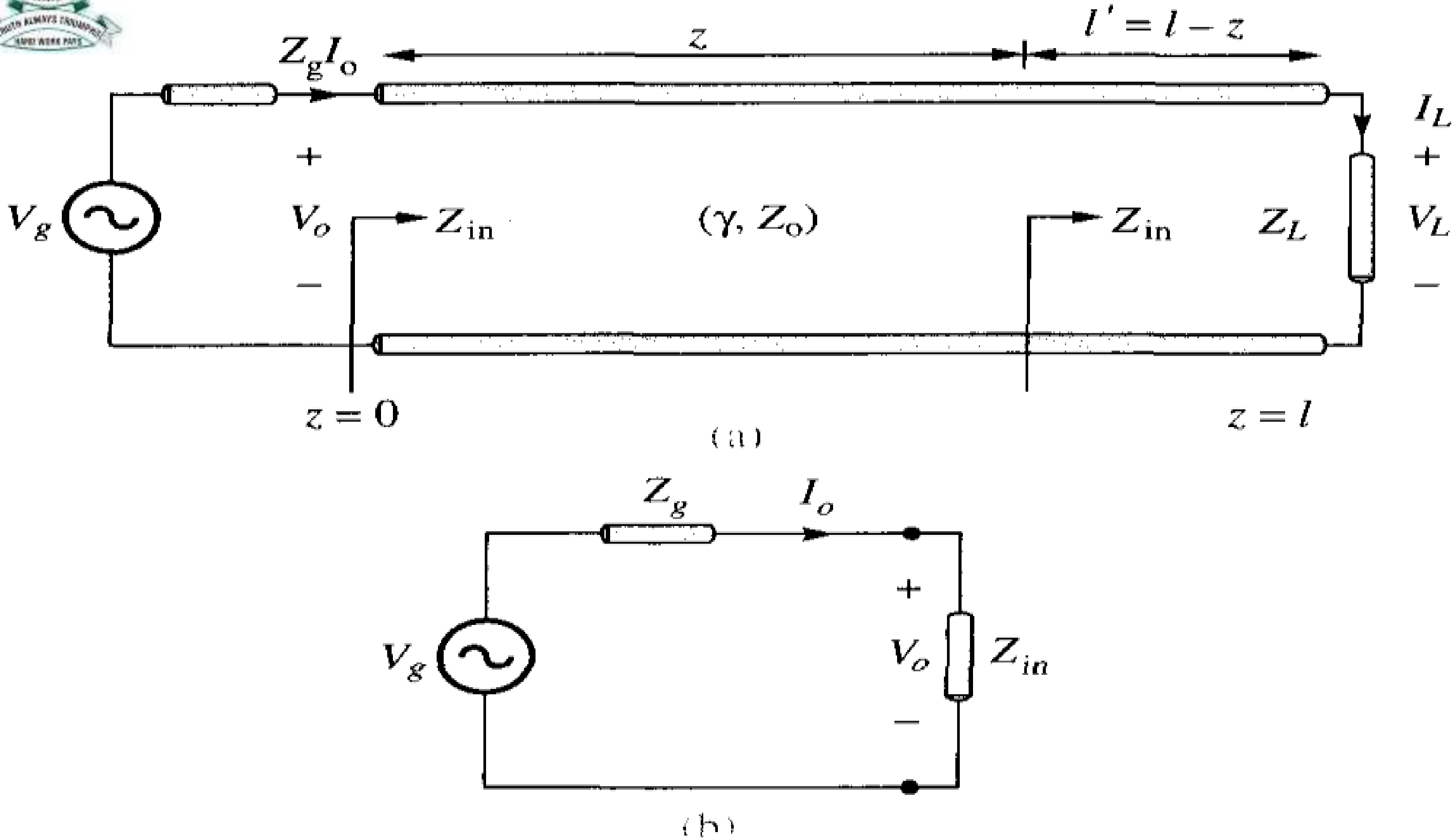


Figure (a) Input impedance due to a line terminated by a load; (b) equivalent circuit for finding V_o and I_o in terms of Z_{in} at the input.

Input impedance of transmission line

Voltage and current equations of transmission line are

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \qquad \text{——— (1)}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \qquad \text{——— (2)}$$

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} \qquad \text{——— (3)}$$

Substitute (3) in (2) ⟶ $I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$ ——— (4)

$$V_o = V(z = 0), \qquad I_o = I(z = 0)$$

At z=0

From (1) $V_0 = V_0^+ + V_0^-$ —— (5) ; At z=0 From (4) $I_0 = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$ ——— (6)

Input impedance of transmission line

$$V_L = V(z = \ell), \qquad I_L = I(z = \ell)$$

$$V_L = V_0^+ e^{-\gamma l} + V_0^- e^{\gamma l} \quad \text{———} \quad (7)$$

$$I_L = I_0^+ e^{-\gamma l} + I_0^- e^{\gamma l} \quad \text{———} \quad (8)$$

$$I_L = \frac{V_0^+}{Z_0} e^{-\gamma l} - \frac{V_0^-}{Z_0} e^{\gamma l}$$

$$I_L Z_0 = V_0^+ e^{-\gamma l} - V_0^- e^{\gamma l} \quad \text{———} \quad (9)$$

(7) + (9) ⟶ $V_0^+ = \frac{1}{2}(V_L + I_L Z_0) e^{\gamma l}$ ——— (10)

(7) - (9) ⟶ $V_0^- = \frac{1}{2}(V_L - I_L Z_0) e^{-\gamma l}$ ——— (11)

Input impedance of transmission line

Input impedance of transmission line $$Z_{in} = \frac{V(z)}{I(z)} \qquad (12)$$

At z=0; substitute (5), (6) in (12) $$Z_{in} = \frac{Z_0(V_0^+ + V_0^-)}{V_0^+ - V_0^-} \qquad (13)$$

Substitute (10), (11) in (13)

$$Z_{in} = \frac{Z_0(\frac{1}{2}(V_L + I_L Z_0)e^{\gamma l} + \frac{1}{2}(V_L - I_L Z_0)e^{-\gamma l})}{(\frac{1}{2}(V_L + I_L Z_0)e^{\gamma l} - \frac{1}{2}(V_L - I_L Z_0)e^{-\gamma l})}$$

$$V_L = I_L Z_L$$

Input impedance of transmission line

$$Z_{in} = \frac{Z_0(\frac{1}{2}(Z_L + Z_0)e^{\gamma l} + \frac{1}{2}(Z_L - Z_0)e^{-\gamma l})}{(\frac{1}{2}(Z_L + Z_0)e^{\gamma l} - \frac{1}{2}(Z_L - Z_0)e^{-\gamma l})}$$

$$Z_{in} = \frac{Z_0(Z_L(\frac{e^{\gamma l} + e^{-\gamma l}}{2}) + Z_0(\frac{e^{\gamma l} - e^{-\gamma l}}{2}))}{(Z_L(\frac{e^{\gamma l} - e^{-\gamma l}}{2}) + Z_0(\frac{e^{\gamma l} + e^{-\gamma l}}{2}))}$$

$$Z_{in} = \frac{Z_0(Z_L \cosh \gamma l + Z_0 \sinh \gamma l)}{(Z_L \sinh \gamma l + Z_0 \cosh \gamma l)}$$

(Lossy line)

$$Z_{in} = Z_0(\frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l})$$

For lossless line $\gamma = \mathrm{j}\beta$

$$\tanh \gamma l = \tanh \mathrm{j}\beta l = j \tan \beta l$$

$$Z_{in} = Z_0(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l})$$

Where βl Is the electrical length of transmission line in radians

Z_{in} Showing that the input impedance varies periodically with distance l from the load

Input impedance of transmission line

Input impedance of transmission line in terms of reflection coefficient

$$Z_{in} = \frac{Z_0(\frac{1}{2}(Z_L + Z_0)e^{\gamma l} + \frac{1}{2}(Z_L - Z_0)e^{-\gamma l})}{(\frac{1}{2}(Z_L + Z_0)e^{\gamma l} - \frac{1}{2}(Z_L - Z_0)e^{-\gamma l})}$$

$$Z_{in} = \frac{Z_0(e^{\gamma l} + (\frac{Z_L - Z_0}{Z_L + Z_0})e^{-\gamma l})}{(e^{\gamma l} - (\frac{Z_L - Z_0}{Z_L + Z_0})e^{-\gamma l})}$$

$$Z_{in} = Z_0(\frac{e^{\gamma l} + \Gamma_L e^{-\gamma l}}{e^{\gamma l} - \Gamma_L e^{-\gamma l}})$$

Reflection coefficient

The voltage reflection coefficient at any on the line is the ratio of the magnitude of the Reflected voltage wave to that of incident wave

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}$$

Incident voltage wave ($V_o^+ e^{-\gamma z}$)   Reflected voltage wave ($V_o^- e^{\gamma z}$)

$$\Gamma(z) = \frac{V_o^- e^{\gamma z}}{V_o^+ e^{-\gamma z}} = \frac{V_o^-}{V_o^+} e^{2\gamma z}$$

At the load $(z = l)$ $$\Gamma_L = \frac{V_o^- e^{\gamma \ell}}{V_o^+ e^{-\gamma \ell}}$$

Reflection coefficient

$$\Gamma_L = \frac{\frac{1}{2}(V_L - I_L Z_0)e^{-\gamma l}e^{\gamma l}}{\frac{1}{2}(V_L + I_L Z_0)e^{\gamma l}e^{-\gamma l}}$$

$$V_L = I_L Z_L$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Current reflection coefficient $= \frac{I_0^- e^{\gamma l}}{I_0^+ e^{-\gamma l}}$

$$= \frac{-\frac{V_0^-}{Z_0} e^{\gamma l}}{\frac{V_0^+}{Z_0} e^{-\gamma l}}$$

$$= -\frac{V_0^- e^{\gamma l}}{V_0^+ e^{-\gamma l}}$$

Current reflection coefficient $= -\Gamma_L$

Current reflection coefficient is negative of voltage reflection coefficient

VSWR

Standing wave ratio is the ratio between maximum voltage (current) and minimum voltage (current) of standing wave

$$S = \frac{V_{\max}}{V_{\min}} = \frac{I_{\max}}{I_{\min}}$$

If standing wave ratio is expressed in terms of voltages then it is called as Voltage Standing Wave Ratio (VSWR)

$$VSWR = \frac{V_{\max}}{V_{\min}}$$

$$V(\mathrm{z}) = V_0^{+} e^{-\gamma z} + V_0^{-} e^{\gamma z}$$

For lossless line $\gamma = \mathrm{j}\beta$; $V(\mathrm{z}) = V_0^{+} e^{-j\beta z} + V_0^{-} e^{j\beta z}$ ———— (1)

VSWR

From (1)

$$\begin{cases} V_{max} = |V_0^+| + |V_0^-| \\ V_{min} = |V_0^+| - |V_0^-| \end{cases}$$

$$VSWR = \frac{V_{max}}{V_{min}} = \frac{|V_0^+| + |V_0^-|}{|V_0^+| - |V_0^-|}$$

$$VSWR = \frac{1 + \left|\frac{V_0^-}{V_0^+}\right|}{1 - \left|\frac{V_0^-}{V_0^+}\right|}$$

Relation between VSWR And Reflection coefficient $\Longrightarrow$

$$\left\{ VSWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \right\}$$

$$\because |\Gamma_L| = \frac{|V_0^- e^{j\beta l}|}{|V_0^+ e^{-j\beta l}|} = \left|\frac{V_0^-}{V_0^+}\right|$$

Range of VSWR and Reflection coefficient

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$$

If $Z_L = 0$ (short circuit) ⟶ $\Gamma_L = -1$

If $Z_L = Z_0$ (Matched condition) ⟶ $\Gamma_L = 0$

If $Z_L = \infty$ (Open circuit) ⟶ $\Gamma_L = 1$

Range of Reflection coefficient ⟶ $-1 \le \Gamma_L \le 1$

$$VSWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

Range of VSWR and Reflection coefficient

$$VSWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

If $\Gamma_L = -1 \longrightarrow VSWR = \infty$

If $\Gamma_L = 0 \longrightarrow VSWR = 1$

If $\Gamma_L = 1 \longrightarrow VSWR = \infty$

Range of VSWR $\longrightarrow 1 < VSWR < \infty$

Short Circuit (SC) Lines

$$Z_{in} = Z_0 \left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right)$$

$$Z_{sc} = Z_{in} \Big|_{Z_L = 0} = jZ_o \tan \beta \ell$$

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$$

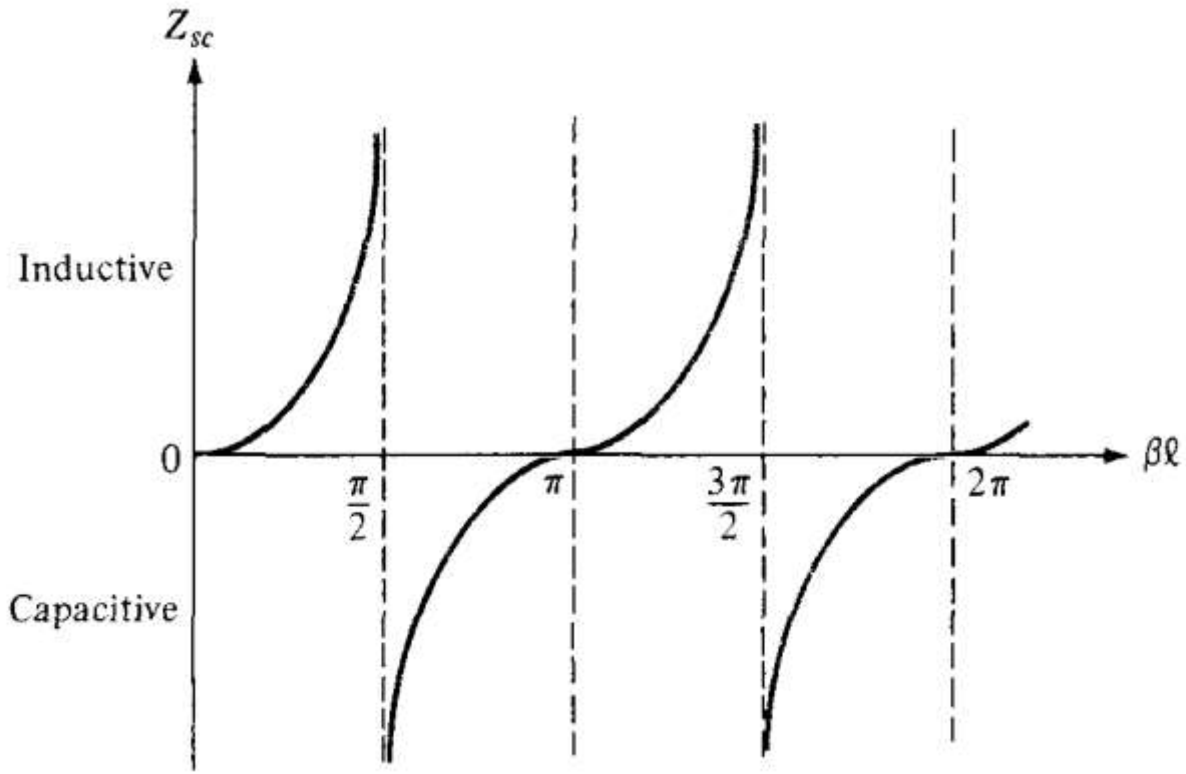
If $Z_L = 0$ (short circuit) → $\Gamma_L = -1$

$$VSWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

If $\Gamma_L = -1$ → $VSWR = \infty$

Short Circuit (SC) Lines

We notice from equation (1) that input impedance of short circuited line is a pure reactance, Which could be capacitive or inductive depending on the value of 'l'. The variation of input impedance of short circuited line with 'l' as shown in figure



Short Circuit (SC) Lines

$$Z_{SC} = jZ_0 \tan \beta l$$

$$\beta l = \frac{\pi}{2}$$

$$\frac{2\pi}{\lambda} l = \frac{\pi}{2}$$

βl	Length of line in terms of λ	Z_{SC}
0	0	0
45^0	λ/8	jZ_0
90^0	λ/4	infinity
135^0	3λ/8	$- jZ_0$
180^0	λ/2	0

Short Circuit (SC) Lines

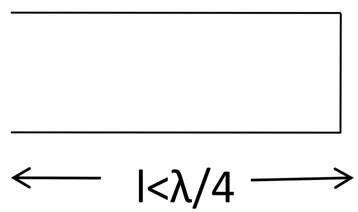


Fig: length of shorted line<λ/4

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Fig: Equivalent circuit of shorted line with l<λ/4

So length of shorted line with length <λ/4 acting as inductor

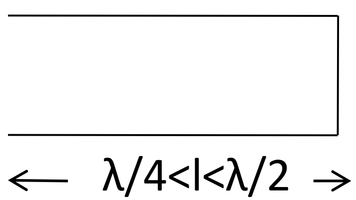

Fig: length of shorted line In between λ/4 and λ/2

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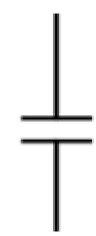

Fig: Equivalent circuit of shorted line with λ/4 <l< λ/2

So length of shorted line In between λ/4 and λ/2 acting as capacitor

Short Circuit (SC) Lines

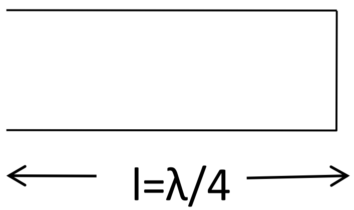

Fig: length of shorted line=λ/4

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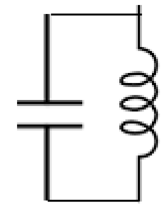


Fig: Equivalent circuit of shorted line with l=λ/4

So length of shorted line with length =λ/4 acting as combination of inductor and capacitor

Open Circuit (OC) Lines

$$Z_{in} = Z_0 \left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right)$$

$$Z_{oc} = \lim_{Z_L \to \infty} Z_{in} = \frac{Z_o}{j \tan \beta \ell} = -jZ_o \cot \beta \ell$$

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$$

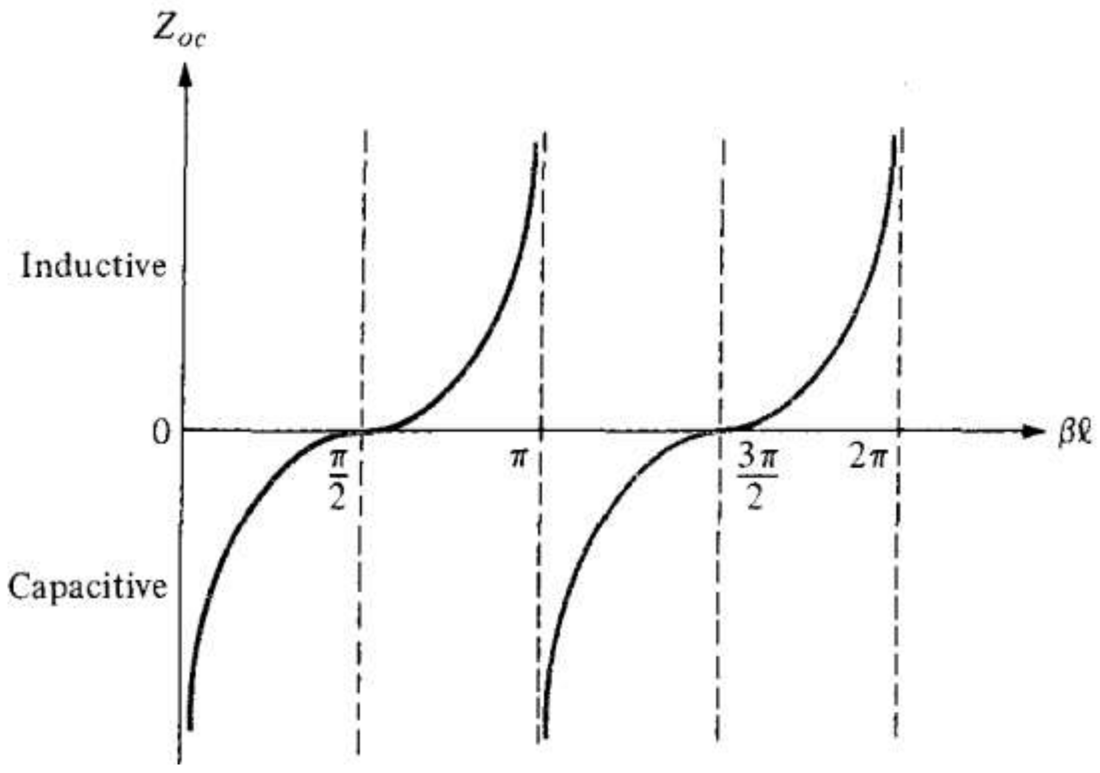
If $Z_L = \infty$ (Open circuit) $\longrightarrow$ $\Gamma_L = 1$

$$VSWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

If $\Gamma_L = 1$ $\longrightarrow$ $VSWR = \infty$

Open Circuit (OC) Lines

We notice from equation (1) that input impedance of short circuited line is a pure reactance, Which could be capacitive or inductive depending on the value of ‘l’. The variation of input impedance of short circuited line with ‘l’ as shown in figure

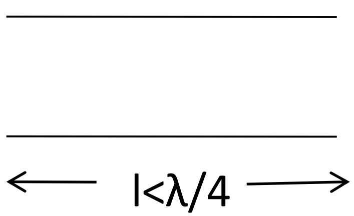


Open Circuit (OC) Lines

$$Z_{OC} = -jZ_0 \cot \beta l$$

βl	Length of line in terms of λ	Z_{OC}
0	0	-infinity
45^0	λ/8	$-jZ_0$
90^0	λ/4	0
135^0	3λ/8	jZ_0
180^0	λ/2	infinity

Open Circuit (OC) Lines



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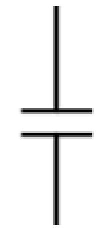

Fig: length of opened line<λ/4

Fig: Equivalent circuit of opened line with l<λ/4

So length of opened line with length <λ/4 acting as capacitor

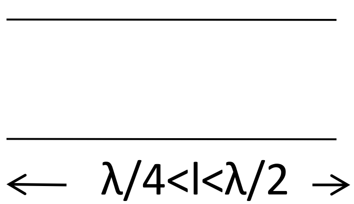

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Fig: length of opened line In between λ/4 and λ/2

Fig: Equivalent circuit of opened line with λ/4 <l< λ/2

So length of opened line In between λ/4 and λ/2 acting as inductor

Open Circuit (OC) Lines

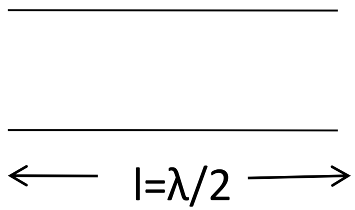

Fig: length of opened line=λ/2

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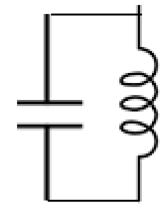


Fig: Equivalent circuit of opened line with l=λ/2

So length of opened line with length =λ/2 acting as combination of inductor and capacitor

$$Z_{SC} = jZ_0 \tan \beta l$$

$$Z_{OC} = -jZ_0 \cot \beta l$$

$$Z_o^{\,2} = Z_{SC} Z_{OC}$$

$$Z_0 = \sqrt{Z_{SC} Z_{OC}}$$

Matched line

This is the most desired case from practical point of view

$$Z_L = Z_0$$

$$Z_{in} = Z_0\left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}\right)$$

$$Z_{in} = Z_o$$

$$\Gamma_L = 0, \qquad s = 1$$

that is, $V_o^- = 0$, the whole wave is transmitted and there is no reflection. The incident power is fully absorbed by the load. Thus maximum power transfer is possible when a transmission line is matched to the load.

Infinite line

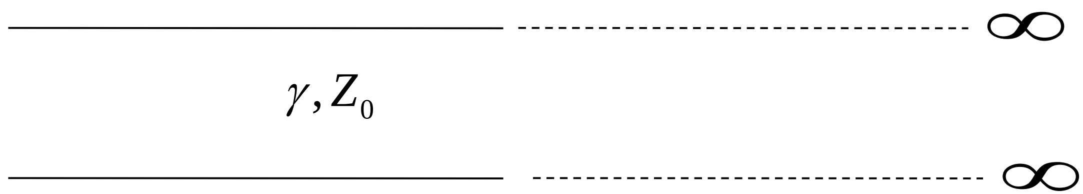


Fig: Infinite line

If a line of infinite length is considered, then all the power fed into it will be absorbed. The Reason being as we move away from the input terminals towards load the current and voltage will decrease along the line and become zero at an infinite distance.

Voltage and current equations of finite transmission line are

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \qquad \text{———} \qquad (1)$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \qquad \text{———} \qquad (2)$$

Infinite line

At infinite distance V(z) = I(z) = 0

From equation (1) $$0 = V_0^{+} e^{-\gamma\infty} + V_0^{-} e^{\gamma\infty} \Rightarrow V_0^{-} = 0$$

From equation (2) $$0 = I_0^{+} e^{-\gamma\infty} + I_0^{-} e^{\gamma\infty} \Rightarrow I_0^{-} = 0$$

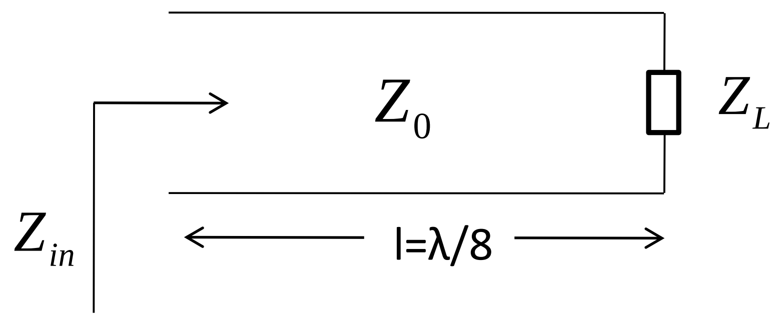
$$V(z) = V_0^{+} e^{-\gamma z} \qquad \text{———} \qquad (3)$$

$$I(z) = I_0^{+} e^{-\gamma z} \qquad \text{———} \qquad (4)$$

So (3), (4) are voltage and current equations of infinite line

From (3), (4) we say that along infinite line there is no reflected wave. So along infinite line all power is terminated without any reflection.

Eighth wave (λ/8) line



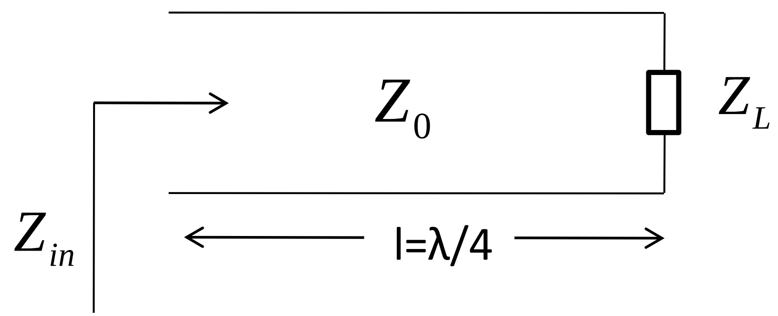
$$Z_{in} = Z_0 \left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right)$$

$$\beta l = \frac{2\pi}{\lambda} \frac{\lambda}{8} = \frac{\pi}{4}$$

$$Z_{in} = Z_0 \left(\frac{Z_L + jZ_0}{Z_0 + jZ_L} \right) \quad ; \quad |Z_{in}| = |Z_0|$$

Significance: Magnitude of Input impedance of eighth wave line is equivalent to line impedance

Quarter wave (λ/4) line



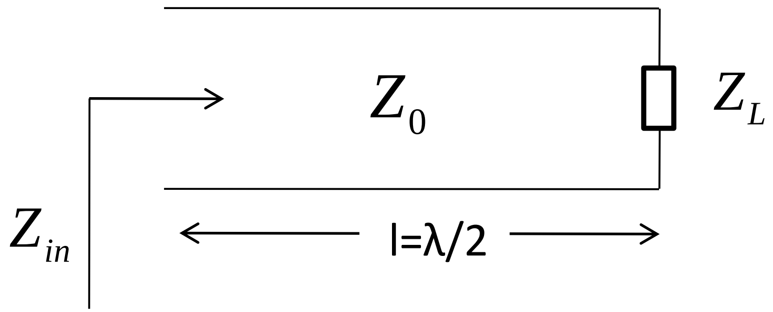
$$Z_{in} = Z_0\left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}\right)$$

$$\beta l = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$$

$$Z_{in} = \frac{Z_0^{\ 2}}{Z_L}$$

Significance: The quarter wave line can transform a low impedance into a high impedance and vice versa, thus it can be considered as an impedance inverter.

Half wave (λ/2) line



$$Z_{in} = Z_0\left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}\right)$$

$$\beta l = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi$$

$$Z_{in} = Z_L$$

Significance: In half wave line load impedance is directly equal to input impedance of line.
So half wave line is considered as one to one transformer or impedance reflector

Power in a transmission line

The voltage and current equations of lossless transmission line at the load are

$$V(z) = V_0^{+} e^{-j\beta l} + V_0^{-} e^{j\beta l} \qquad \text{———} \quad (1)$$

$$I(z) = I_0^{+} e^{-j\beta l} + I_0^{-} e^{j\beta l} \qquad \text{———} \quad (2)$$

From (1)

$$V(z) = V_0^{+} e^{-j\beta l} \left(1 + \frac{V_0^{-} e^{j\beta l}}{V_0^{+} e^{-j\beta l}}\right)$$

$$V(z) = V_0^{+} e^{-j\beta l} (1 + \Gamma_L)$$

From (2)

$$I(z) = I_0^{+} e^{-j\beta l} \left(1 + \frac{I_0^{-} e^{j\beta l}}{I_0^{+} e^{-j\beta l}}\right)$$

Power in a transmission line

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta l}(1-\Gamma_L)$$

$$I^*(z) = \frac{V_0^{+*}}{Z_0} e^{j\beta l}(1-\Gamma_L^*)$$

$$\mathrm{P}_{avg} = \frac{1}{2}\mathrm{Re}(V(z)I^*(\mathrm{z}))$$

$$\mathrm{P}_{avg} = \frac{1}{2}\mathrm{Re}(V_0^+ e^{-j\beta l}(1+\Gamma_L)\frac{V_0^{+*}}{Z_0} e^{j\beta l}(1-\Gamma_L^*))$$

$$\left[\quad P_{avg} = \frac{|V_0^+|^2}{2Z_0}(1-|\Gamma_L|^2)\quad\right]$$

Power in a transmission line

$$P_{avg} = \frac{|V_0^+|^2}{2Z_0}(1 - |\Gamma_L|^2) \qquad \text{———————} \qquad (3)$$

$$P_t = P_i - P_r \qquad ; \qquad P_t = P_{avg}$$

From (3) first term is the incident power Pi while the second term is the reflected power Pr
From (3) we should notice that maximum power is delivered to the load when $\Gamma_L = 0$

Problems

A 30-m-long lossless transmission line with $Z_o = 50\ \Omega$ operating at 2 MHz is terminated with a load $Z_L = 60 + j40\ \Omega$. If $u = 0.6c$ on the line, find

(a) The reflection coefficient Γ

(b) The standing wave ratio s

(c) The input impedance

This problem will be solved with and without using the Smith chart.

Method 1: (Without the Smith chart)

(a) $$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{60 + j40 - 50}{50 + j40 + 50} = \frac{10 + j40}{110 + j40}$$
$$= 0.3523\underline{/56^\circ}$$

(b) $$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.3523}{1 - 0.3523} = 2.088$$

Problems

(c) Since $u = \omega/\beta$, or $\beta = \omega/u$,

$$\beta\ell = \frac{\omega\ell}{u} = \frac{2\pi\,(2 \times 10^6)(30)}{0.6\,(3 \times 10^8)} = \frac{2\pi}{3} = 120^\circ$$

Note that $\beta\ell$ is the electrical length of the line.

$$\begin{aligned} Z_{in} &= Z_o\left[\frac{Z_L + jZ_o \tan\beta\ell}{Z_o + jZ_L \tan\beta\ell}\right] \\ &= \frac{50\,(60 + j40 + j50\tan 120^\circ)}{[50 + j(60 + j40)\tan 120^\circ]} \\ &= \frac{50\,(6 + j4 - j5\sqrt{3})}{(5 + 4\sqrt{3} - j6\sqrt{3})} = 24.01\,\underline{/3.22^\circ} \\ &= 23.97 + j1.35\ \Omega \end{aligned}$$

Smith Chart

Graphical indication of impedance of transmission line as one moves along line

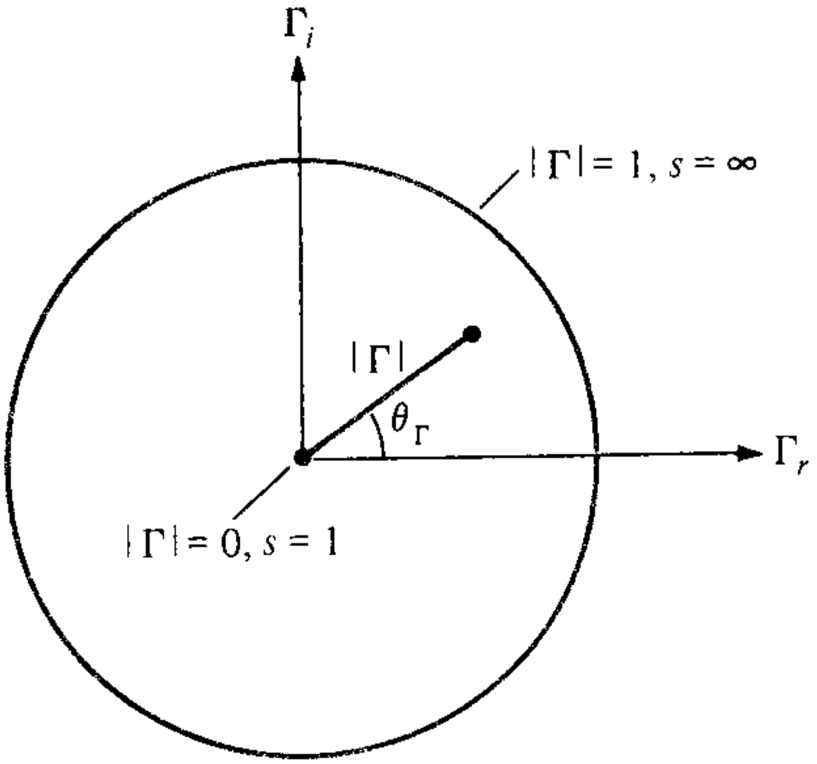


Fig: Unit Circle on which Smith chart is constructed

Smith Chart

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o}$$

$$Z = R + jX$$

$$z_L = \frac{Z_L}{Z_o} = r + jx$$

$$\Gamma = |\Gamma| \angle \theta_\Gamma = \Gamma_r + j\Gamma_i$$

where Γ_r and Γ_i are the real and imaginary parts of the reflection coefficient Γ

$$\Gamma = \Gamma_r + j\Gamma_i = \frac{z_L - 1}{z_L + 1}$$

Smith Chart

$$\Gamma = \Gamma_r + j\Gamma_i = \frac{z_L - 1}{z_L + 1}$$

$$z_L = r + jx = \frac{(1 + \Gamma_r) + j\Gamma_i}{(1 - \Gamma_r) - j\Gamma_i}$$

$$r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \qquad \text{———————— (1)}$$

$$x = \frac{2\,\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2} \qquad \text{———————— (2)}$$

Smith Chart

From (1)

$$\left[\Gamma_r - \frac{r}{1+r}\right]^2 + \Gamma_i^2 = \left[\frac{1}{1+r}\right]^2$$

$$\text{center at } (\Gamma_r, \Gamma_i) = \left(\frac{r}{1+r}, 0\right)$$

$$\text{radius} = \frac{1}{1+r}$$

Smith Chart

TABLE Radii and Centers of r-Circles for Typical Values of r

Normalized Resistance (r)	Radius $\left(\frac{1}{1+r}\right)$	Center $\left(\frac{r}{1+r}, 0\right)$
0	1	(0, 0)
1/2	2/3	(1/3, 0)
1	1/2	(1/2, 0)
2	1/3	(2/3, 0)
5	1/6	(5/6, 0)
∞	0	(1, 0)

Smith Chart

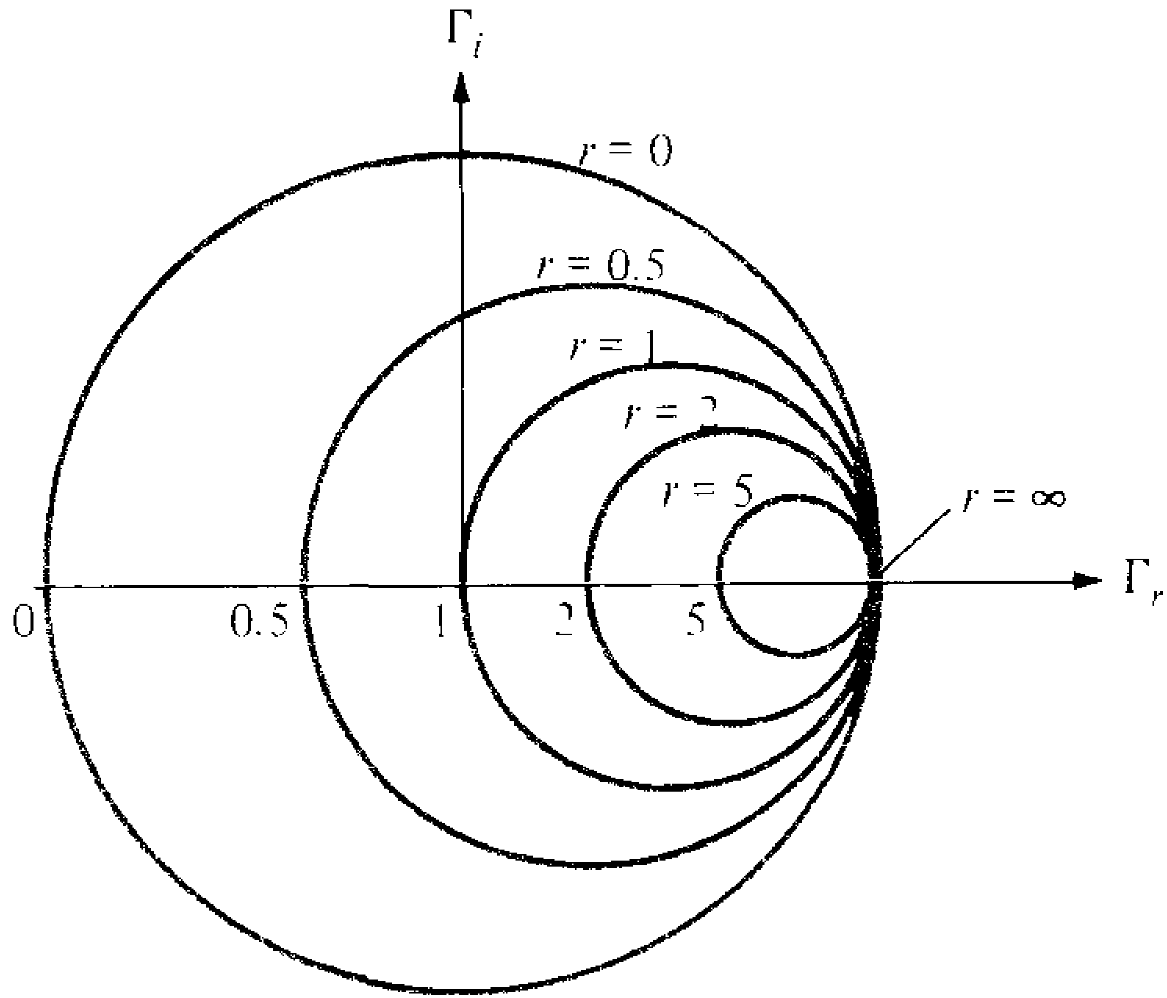


Figure Typical *r*-circles for $r = 0, 0.5, 1, 2, 5, \infty$.

Smith Chart

From (2)

$$[\Gamma_r - 1]^2 + \left[\Gamma_i - \frac{1}{x}\right]^2 = \left[\frac{1}{x}\right]^2$$

$$\text{center at } (\Gamma_r, \Gamma_i) = \left(1, \frac{1}{x}\right)$$

$$\text{radius} = \frac{1}{x}$$

Smith Chart

TABLE Radii and Centers of x-Circles for Typical Value of x

Normalized Reactance (x)	Radius $\left(\frac{1}{x}\right)$	Center $\left(1, \frac{1}{x}\right)$
0	∞	$(1, \infty)$
$\pm 1/2$	2	$(1, \pm 2)$
± 1	1	$(1, \pm 1)$
± 2	1/2	$(1, \pm 1/2)$
± 5	1/5	$(1, \pm 1/5)$
$\pm \infty$	0	$(1, 0)$

Smith Chart

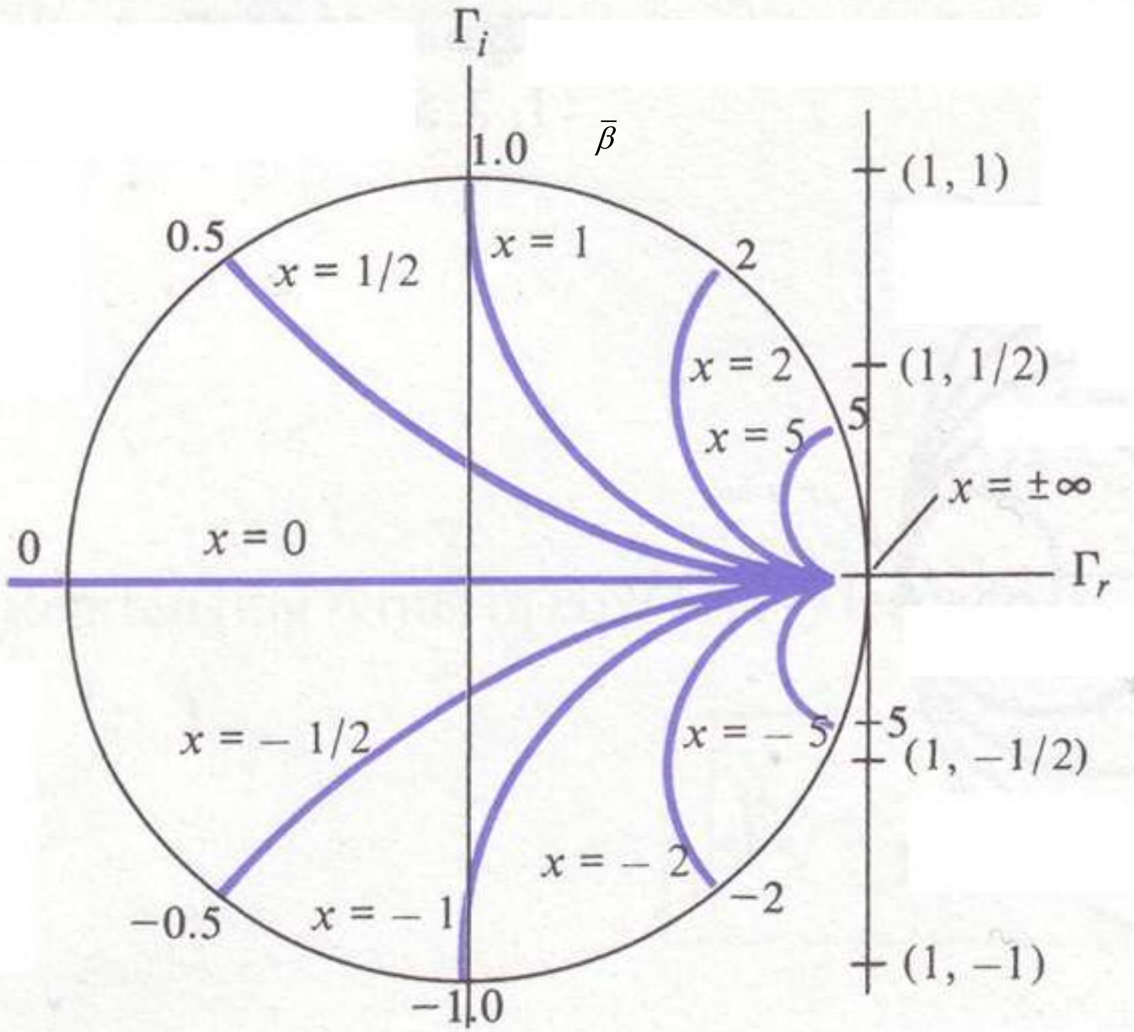
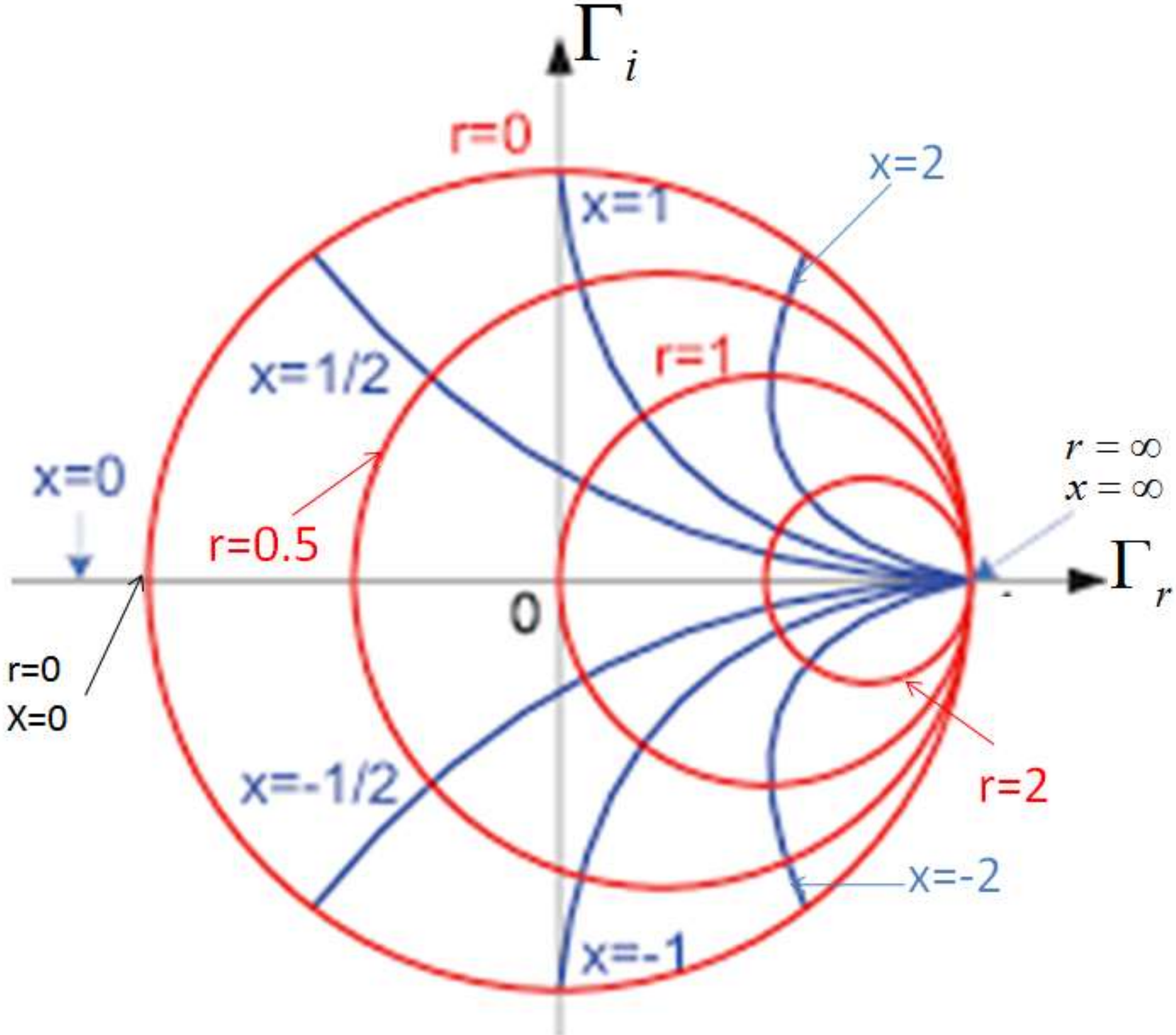
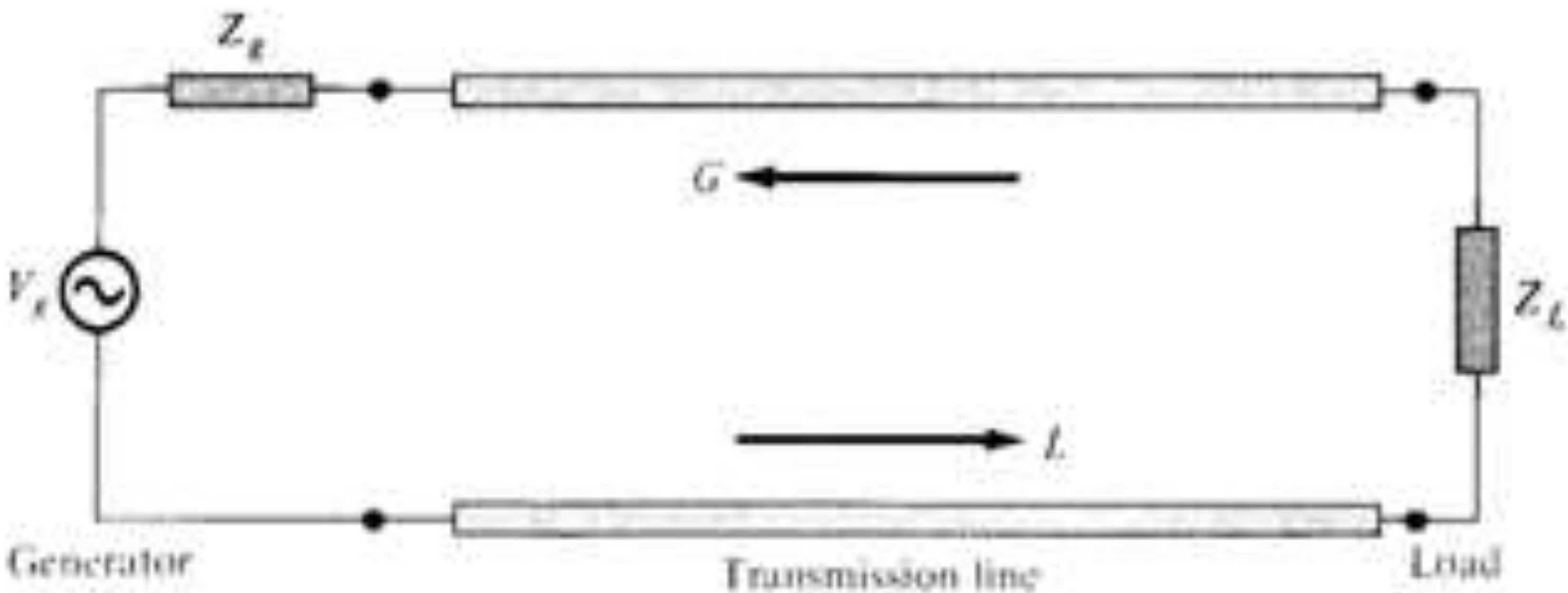


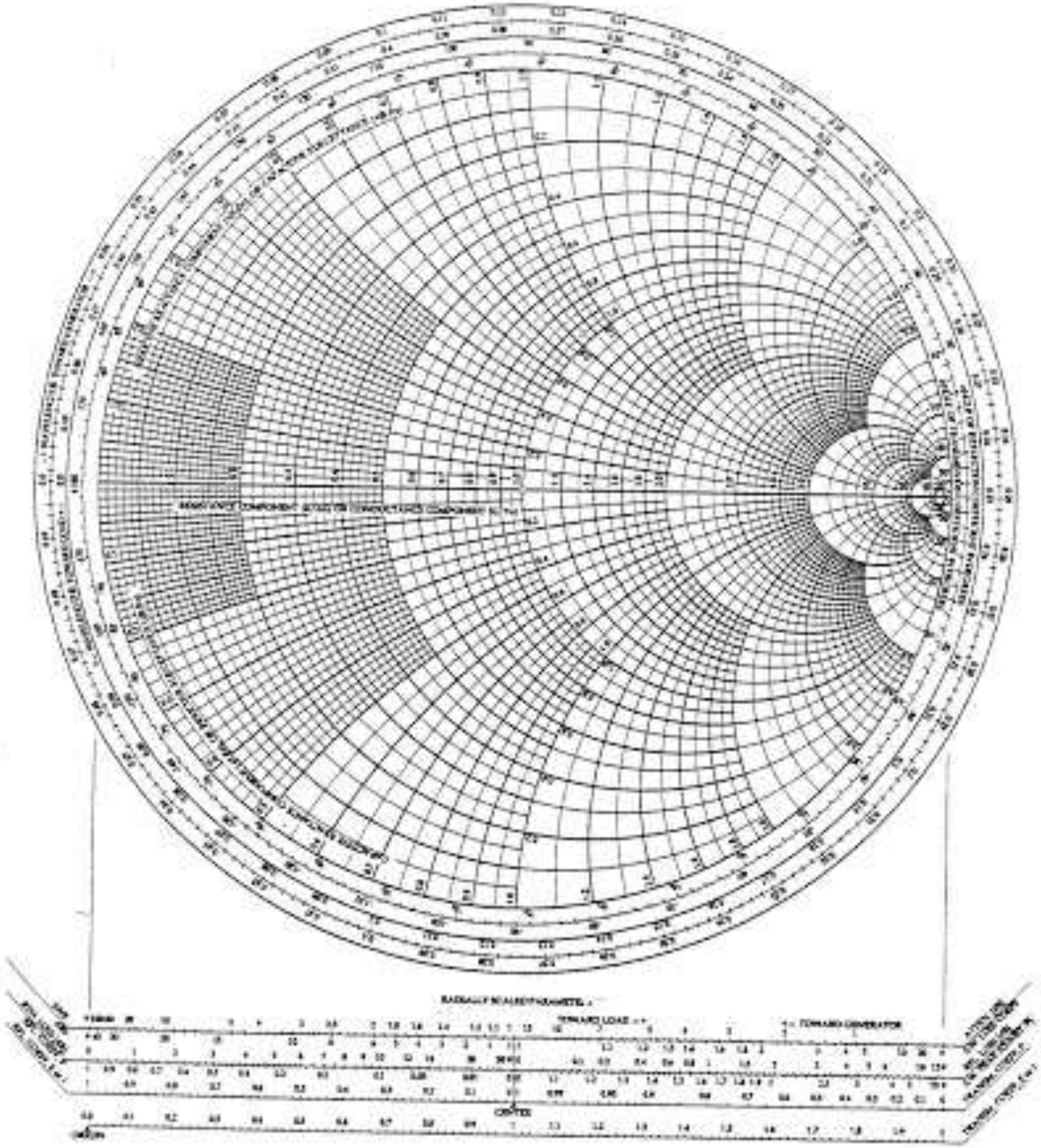
Figure Typical x-circles for $x = 0, \pm 1/2, \pm 1, \pm 2, \pm 5, \pm\infty$.



Γi
r=0
x=2
x=1
r=1
x=1/2
r = ∞
x = ∞
x=0
r=0.5
Γr
0
r=0
X=0
x=-1/2
r=2
x=-2
x=-1

Smith Chart




TOWARD LOAD
CENTER

Smith Chart

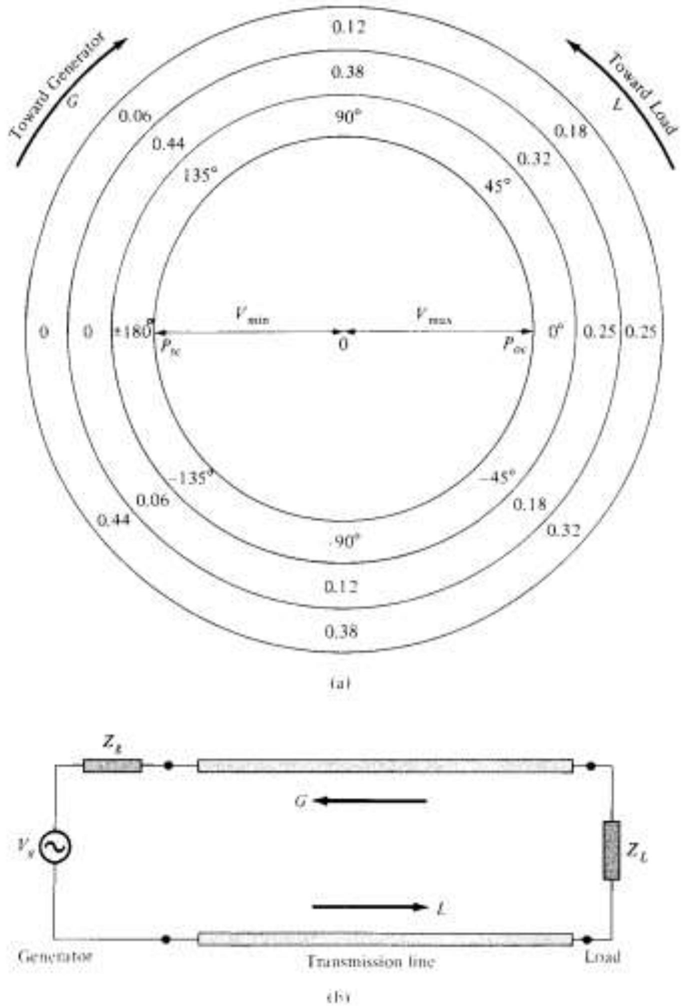
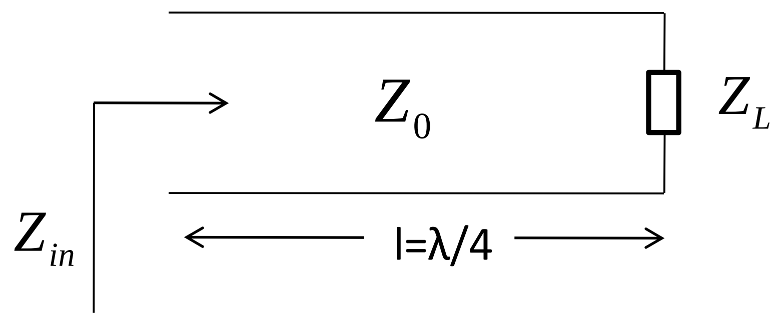


Figure (a) Smith chart illustrating scales around the periphery and movements around the chart, (b) corresponding movements along the transmission line.

Quarter wave transformer

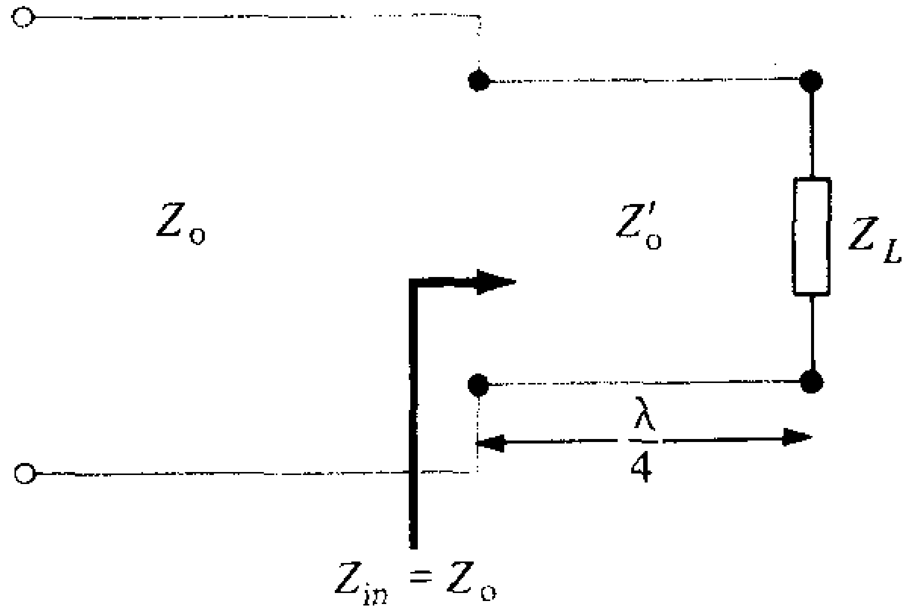


$$Z_{in} = Z_0\left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}\right)$$

$$\beta l = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$$

$$Z_{in} = \frac{{Z_0}^2}{Z_L}$$

Quarter wave transformer



$$Z'_0 = \sqrt{Z_0 Z_L}$$

$$Z_{in} = \frac{Z'^2_0}{Z_L}$$

$$Z_{in} = \frac{Z_0 Z_L}{Z_L}$$

$$Z_{in} = Z_0$$

Quarter wave transformer

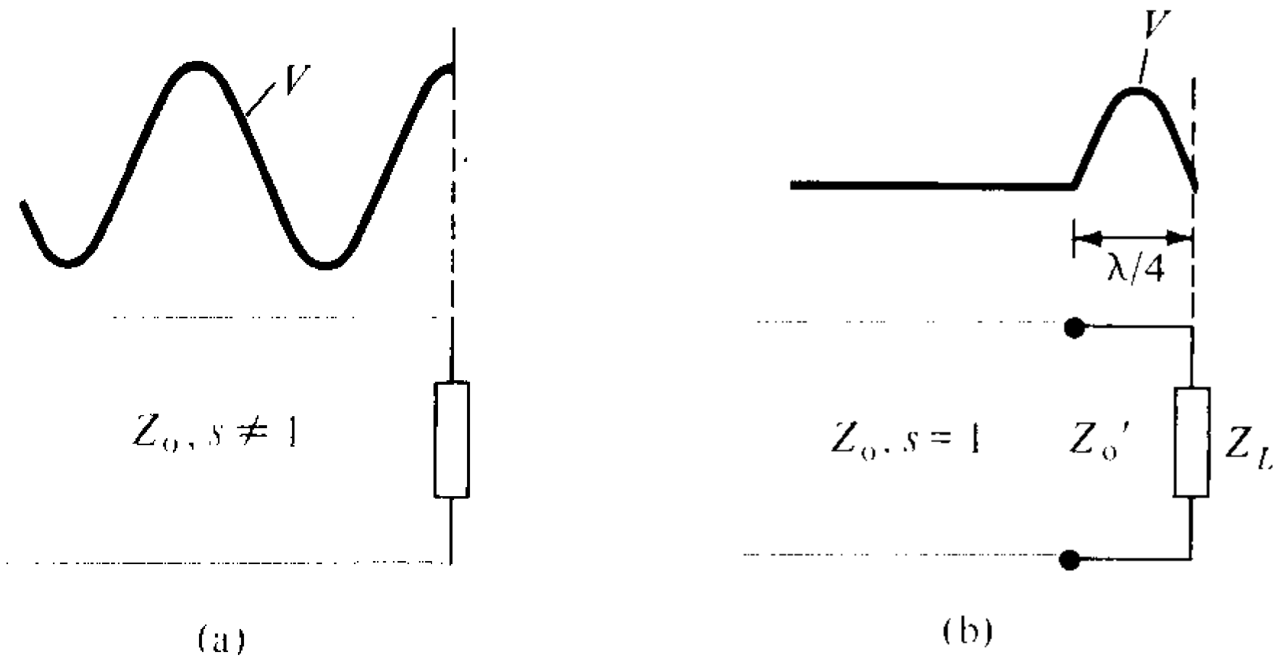


Figure Voltage standing-wave pattern of mismatched load: **(a)** without a λ/4 transformer, **(b)** with a λ/4 transformer.

Stub Matching

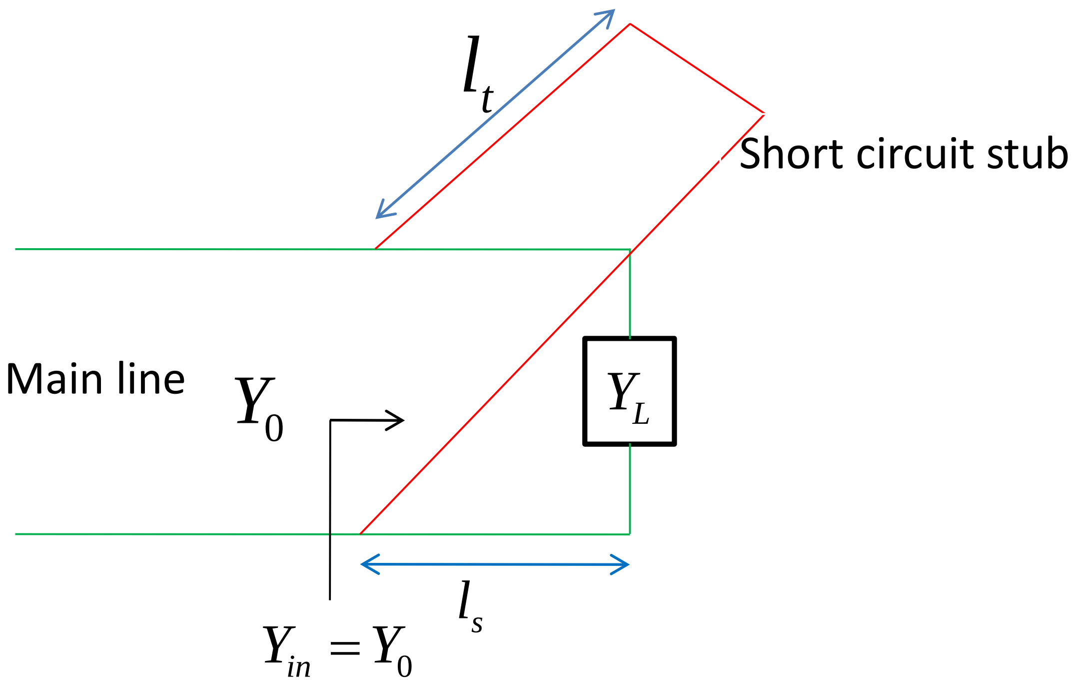


Fig: Transmission line with single stub

Short circuit stubs are Preferable than open circuit Stubs because

(i)Infinite terminating impedance is more difficult to realize than zero terminating impedance

(ii)Radiation is more in open circuit Stubs

A use of open or closed stub line of suitable length as a reactance shunted across the transmission line at a suitable distance from the load to tune the line and the load is called as stub matching

Stub Matching

$$Z_{in} = Z_0 \left(\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right)$$

$$Y_{in} = \frac{1}{Z_{in}} = \frac{Z_0 + jZ_L \tan \beta l}{Z_0 (Z_L + jZ_0 \tan \beta l)}$$

$$Y_{in} = Y_0 \left(\frac{Y_L + jY_o \tan \beta l}{Y_0 + jY_L \tan \beta l} \right)$$

$$\frac{Y_{in}}{Y_0} = y_{in} \leftarrow$$ Normalized input admittance

$$\frac{Y_L}{Y_0} = y_L \leftarrow$$ Normalized Load admittance

Stub Matching

$$y_{in} = \frac{y_L + j\tan\beta l}{1 + jy_L \tan\beta l}$$

$$y_{in} = \frac{y_L(1+\tan^2\beta l) + j\tan\beta l(1-{y_L}^2)}{1+{y_L}^2\tan^2\beta l}$$

$$y_{in} = g_{in} + jb_{in}$$

$$g_{in} = \frac{y_L(1+\tan^2\beta l)}{1+{y_L}^2\tan^2\beta l}$$

$$b_{in} = \frac{\tan\beta l(1-{y_L}^2)}{1+{y_L}^2\tan^2\beta l}$$

Stub Matching

$$y_{in} = g_{in} + jb_{in} = 1 + j0$$

$$\frac{Y_{in}}{Y_0} = 1$$

$$Y_{in} = Y_0$$

$$Z_{in} = Z_0$$

$$g_{in} = 1$$

$$b_{in} = 0$$

$$g_{in} = 1 = \frac{y_L(1 + \tan^2 \beta l)}{1 + y_L{}^2 \tan^2 \beta l}$$

$$1 + y_L{}^2 \tan^2 \beta l = y_L + y_L \tan^2 \beta l$$

$$y_L \tan^2 \beta l_s (y_L - 1) = y_L - 1$$

Stub Matching

$$\tan^2 \beta l_s = \frac{1}{y_L} = \frac{Y_0}{Y_L}$$

Position of stub from the load $\left[l_s = \frac{\lambda}{2\pi} Tan^{-1} \sqrt{\frac{Z_L}{Z_0}} \right]$

$$b_{in} = \frac{\tan \beta l_s (1 - y_L{}^2)}{1 + y_L{}^2 \tan^2 \beta l_s}$$

$$\frac{B_{in}}{Y_0} = \frac{(1 - \frac{Y_L{}^2}{Y_0{}^2})\sqrt{\frac{Y_0}{Y_L}}}{1 + \frac{Y_L{}^2}{Y_0{}^2} \frac{Y_0}{Y_L}}$$

Stub Matching

$$\frac{B_{in}}{Y_0} = \sqrt{\frac{Y_0}{Y_L}}(1 - \frac{Y_L}{Y_0})$$

$$B_{in} = \sqrt{\frac{Y_0}{Y_L}}(Y_0 - Y_L)$$

$$Z_{SC} = jZ_0 \tan \beta l_t$$

$$Y_{SC} = G_{SC} + jB_{SC} = -jY_0 \cot \beta l_t$$

$$B_{SC} = -jY_0 \cot \beta l_t$$

$$B_{in} + B_{SC} = 0$$

Stub Matching

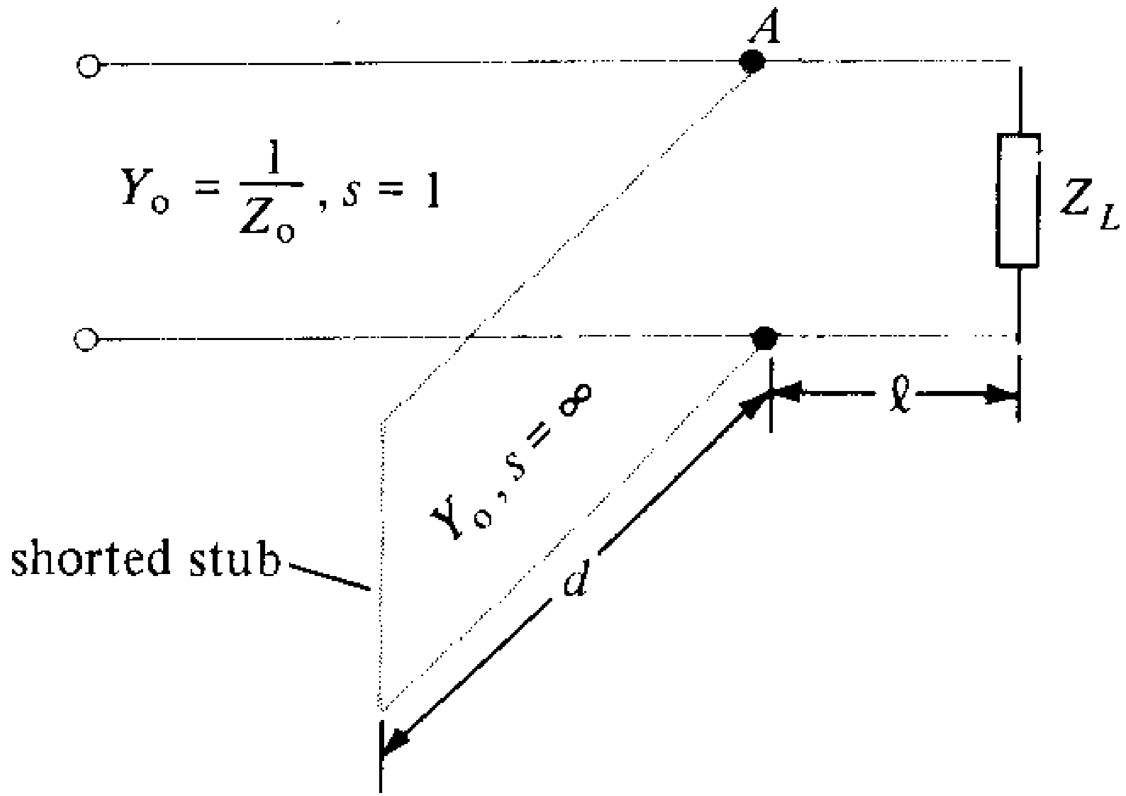
$$\sqrt{\frac{Y_0}{Y_L}}(Y_0 - Y_L) - jY_0 \cot \beta l_t$$

$$\tan \beta l_t = \frac{\sqrt{Y_0 Y_L}}{Y_0 - Y_L}$$

$$l_t = \frac{\lambda}{2\pi} Tan^{-1} \frac{\sqrt{Y_0 Y_L}}{Y_0 - Y_L}$$

Length of stub $\left\{ \left[l_t = \frac{\lambda}{2\pi} Tan^{-1} \frac{\sqrt{Z_L Z_0}}{Z_L - Z_0} \right] \right.$

Stub Matching



$$y_{in} = 1 + jb + y_s = 1 + jb - jb = 1 + j0$$

Questions

1.Discuss the variation of input impedance with electrical length of transmission line for different loads.

2.Determine the reflection coefficients when
(i)$Z_L=Z_0$ (ii)Z_L=short circuit (iii) Z_L=open circuit (iv)Z_L=purely reactive

3.Interpret the impedance matching with the use of quarter wave transformer.

4.Develop the input impedance of transmission line from transmission line equations.

5.Summarize the significance of $\lambda/8$, $\lambda/4$, $\lambda/2$ lines.

6.Interpret the impedance matching with the use of stub matching.

7.Interpret how Smith chart is useful in solving transmission line problems compared with analytical approach.

8.Derive the relation between reflection coefficient and VSWR.

9.Describe how UHF lines can be treated as circuit elements using the necessary equivalent circuits.

Unit-2
Transmission Lines-II

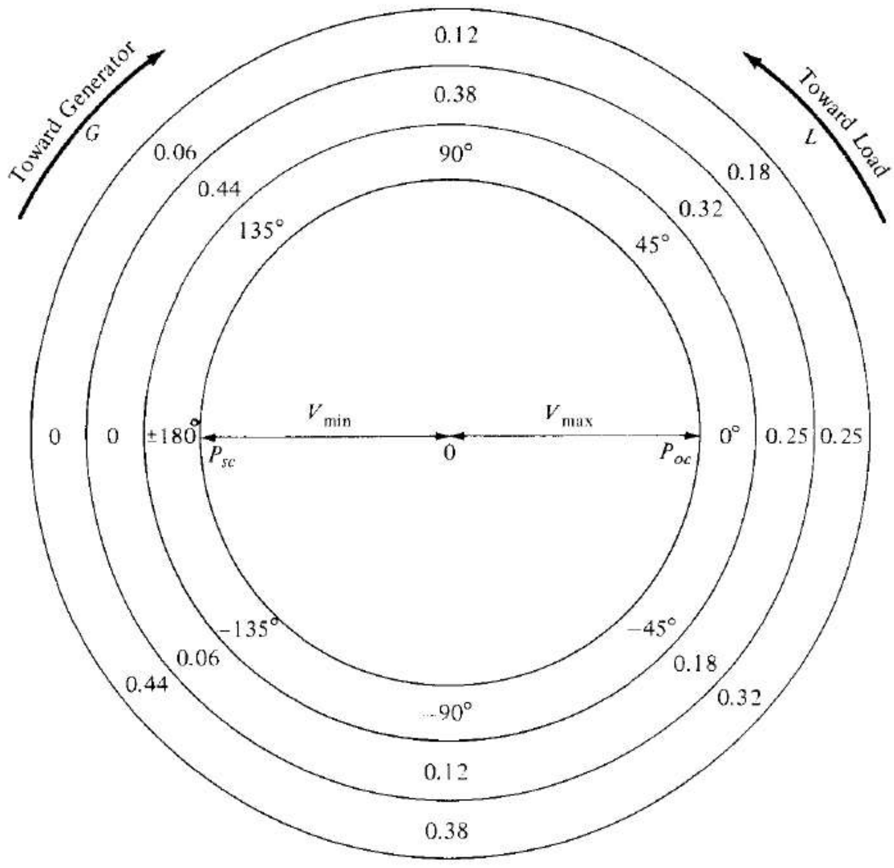
END of UNIT-II

The following points should be noted about the Smith chart:

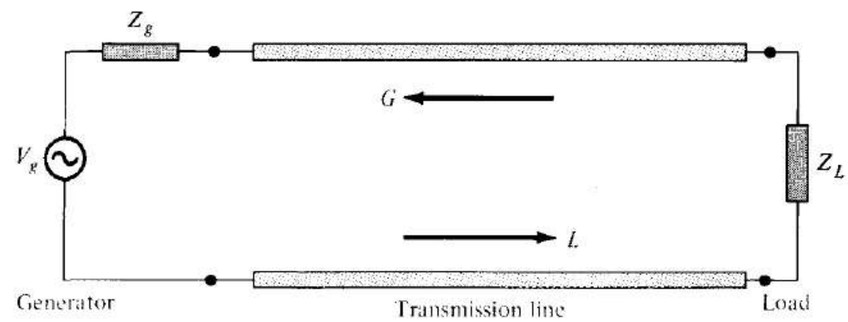
1. At point P_{sc} on the chart $r = 0, x = 0$; that is, $Z_L = 0 + j0$ showing that P_{sc} represents a short circuit on the transmission line. At point P_{oc}, $r = \infty$ and $x = \infty$, or $Z_L = \infty + j\infty$, which implies that P_{oc} corresponds to an open circuit on the line. Also at P_{oc}, $r = 0$ and $x = 0$, showing that P_{oc} is another location of a short circuit on the line.
2. A complete revolution (360°) around the Smith chart represents a distance of $\lambda/2$ on the line. Clockwise movement on the chart is regarded as moving toward the generator (or away from the load) as shown by the arrow G in Figure 11.14(a) and (b). Similarly, counterclockwise movement on the chart corresponds to moving toward the load (or away from the generator) as indicated by the arrow L in Figure 11.14. Notice from Figure 11.14(b) that at the load, moving toward the load does not make sense (because we are already at the load). The same can be said of the case when we are at the generator end.
3. There are three scales around the periphery of the Smith chart as illustrated in Figure 11.14(a). The three scales are included for the sake of convenience but they are actually meant to serve the same purpose; one scale should be sufficient. The scales are used in determining the distance from the load or generator in degrees or wavelengths. The outermost scale is used to determine the distance on the line from the generator end in terms of wavelengths, and the next scale determines the distance from the load end in terms of wavelengths. The innermost scale is a protractor (in degrees) and is primarily used in determining θ_Γ; it can also be used to determine the distance from the load or generator. Since a $\lambda/2$ distance on the line corresponds to a movement of 360° on the chart, λ *distance on the line corresponds to a 720° movement on the chart.*

$$\boxed{\lambda \rightarrow 720^\circ} \tag{11.55}$$

   Thus we may ignore the other outer scales and use the protractor (the innermost scale) for all our θ_Γ and distance calculations.
4. V_{max} occurs where $Z_{in,max}$ is located on the chart [see eq. (11.39a)], and that is on the positive Γ_r axis or on OP_{oc} in Figure 11.14(a). V_{min} is located at the same point where we have $Z_{in,min}$ on the chart; that is, on the negative Γ_r axis or on OP_{sc} in Figure 11.14(a). Notice that V_{max} and V_{min} (or $Z_{in,max}$ and $Z_{in,min}$) are $\lambda/4$ (or 180°) apart.
5. The Smith chart is used both as impedance chart and admittance chart ($Y = 1/Z$). As admittance chart (normalized impedance $y = Y/Y_o = g + jb$), the g- and b-circles correspond to r- and x-circles, respectively.



(a)



(b)

Figure Smith chart illustrating scales around the periphery and movements around the chart, **(b)** corresponding movements along the transmission line.

Based on these important properties, the Smith chart may be used to determine, among other things, (a) $\Gamma = |\Gamma|\underline{/\theta_\Gamma}$ and s; (b) Z_{in} or Y_{in}; and (c) the locations of V_{max} and V_{min} provided that we are given Z_o, Z_L, and the length of the line. Some examples will clearly show how we can do all these and much more with the aid of the Smith chart, a compass, and a plain straightedge.

EXAMPLE 11.4 A 30-m-long lossless transmission line with $Z_o = 50\ \Omega$ operating at 2 MHz is terminated with a load $Z_L = 60 + j40\ \Omega$. If $u = 0.6c$ on the line, find

(a) The reflection coefficient Γ

(b) The standing wave ratio s

(c) The input impedance

Solution:

This problem will be solved with and without using the Smith chart.

Method 1: (Without the Smith chart)

(a) $$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{60 + j40 - 50}{50 + j40 + 50} = \frac{10 + j40}{110 + j40}$$
$$= 0.3523\underline{/56^\circ}$$

(b) $$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.3523}{1 - 0.3523} = 2.088$$

(c) Since $u = \omega/\beta$, or $\beta = \omega/u$,

$$\beta\ell = \frac{\omega\ell}{u} = \frac{2\pi\,(2 \times 10^6)(30)}{0.6\,(3 \times 10^8)} = \frac{2\pi}{3} = 120^\circ$$

Note that $\beta\ell$ is the electrical length of the line.

$$Z_{\text{in}} = Z_o \left[\frac{Z_L + jZ_o \tan \beta\ell}{Z_o + jZ_L \tan \beta\ell}\right]$$
$$= \frac{50\,(60 + j40 + j50 \tan 120^\circ)}{[50 + j(60 + j40) \tan 120^\circ]}$$
$$= \frac{50\,(6 + j4 - j5\sqrt{3})}{(5 + 4\sqrt{3} - j6\sqrt{3})} = 24.01\underline{/3.22^\circ}$$
$$= 23.97 + j1.35\ \Omega$$

Method 2: (Using the Smith chart).

(a) Calculate the normalized load impedance

$$z_L = \frac{Z_L}{Z_o} = \frac{60 + j40}{50}$$
$$= 1.2 + j0.8$$

Locate z_L on the Smith chart of Figure 11.15 at point P where the $r = 1.2$ circle and the $x = 0.8$ circle meet. To get Γ at z_L, extend OP to meet the $r = 0$ circle at Q and measure OP and OQ. Since OQ corresponds to $|\Gamma| = 1$, then at P,

$$|\Gamma| = \frac{OP}{OQ} = \frac{3.2\text{ cm}}{9.1\text{ cm}} = 0.3516$$

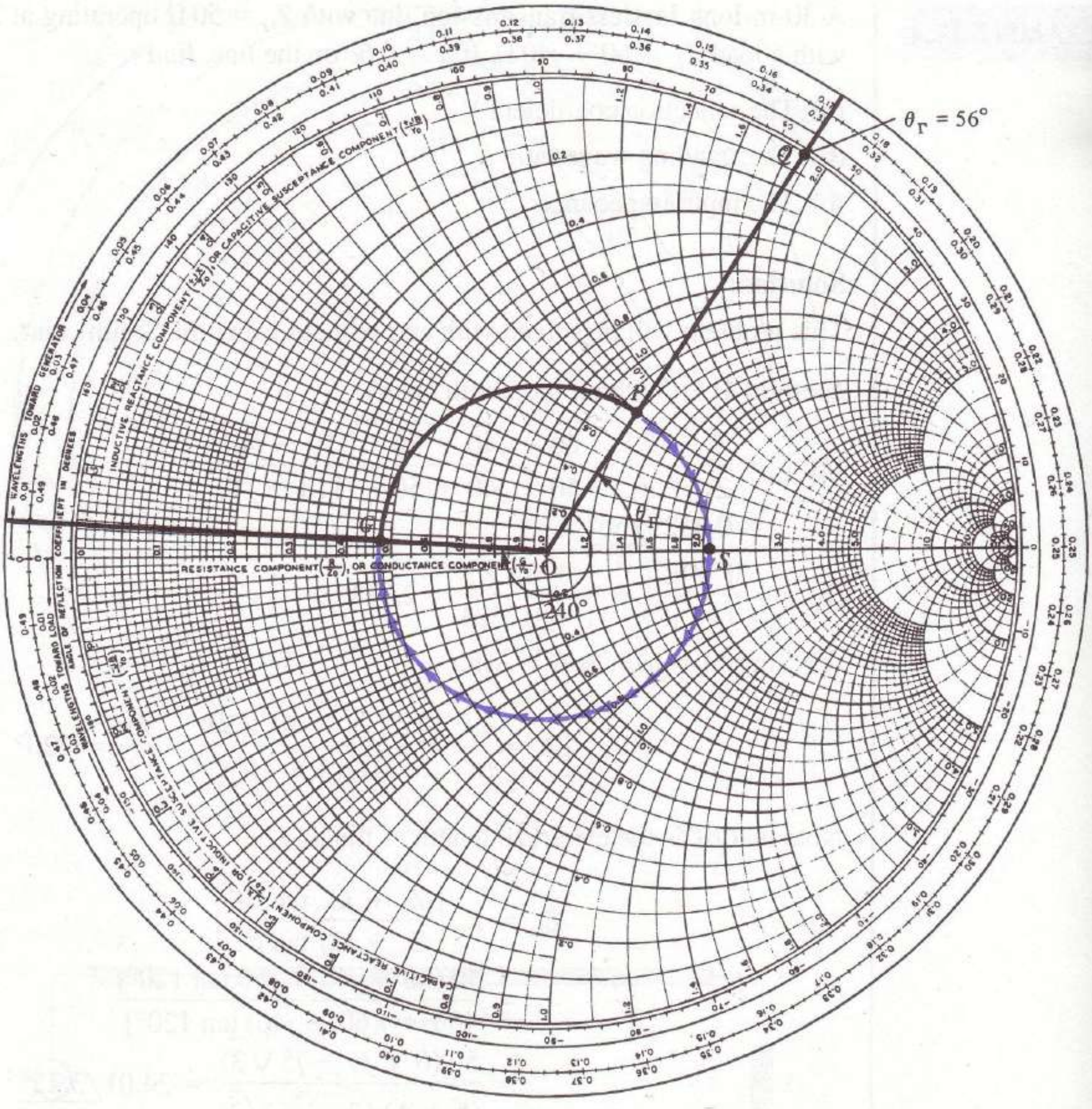


Figure 11.15 For Example 11.4.

Note that $OP = 3.2$ cm and $OQ = 9.1$ cm were taken from the Smith chart used by the author; the Smith chart in Figure 11.15 is reduced but the ratio of OP/OQ remains the same.

Angle θ_Γ is read directly on the chart as the angle between OS and OP; that is

$$\theta_\Gamma = \text{angle } POS = 56^\circ$$

Thus

$$\Gamma = 0.3516 \, \underline{/56^\circ}$$

(b) To obtain the standing wave ratio s, draw a circle with radius OP and center at O. This is the constant s or $|\Gamma|$ circle. Locate point S where the s-circle meets the Γ_r-axis.

[This is easily shown by setting $\Gamma_i = 0$ in eq. (11.49a).] The value of r at this point is s; that is

$$s = r \text{ (for } r \geq 1) \\ = 2.1$$

(c) To obtain Z_{in}, first express ℓ in terms of λ or in degrees.

$$\lambda = \frac{u}{f} = \frac{0.6\,(3 \times 10^8)}{2 \times 10^6} = 90 \text{ m}$$

$$\ell = 30 \text{ m} = \frac{30}{90}\lambda = \frac{\lambda}{3} \rightarrow \frac{720°}{3} = 240°$$

Since λ corresponds to an angular movement of 720° on the chart, the length of the line corresponds to an angular movement of 240°. That means we move toward the generator (or away from the load, in the clockwise direction) 240° on the s-circle from point P to point G. At G, we obtain

$$z_{in} = 0.47 + j0.035$$

Hence

$$Z_{in} = Z_o z_{in} = 50(0.47 + j0.035) = 23.5 + j1.75\ \Omega$$

Although the results obtained using the Smith chart are only approximate, for engineering purposes they are close enough to the exact ones obtained in Method 1.

PRACTICE EXERCISE 11.4

A 70-Ω lossless line has $s = 1.6$ and $\theta_\Gamma = 300°$. If the line is 0.6λ long, obtain

(a) Γ, Z_L, Z_{in}

(b) The distance of the first minimum voltage from the load

Answer: (a) $0.228\ \underline{/300°}$, $80.5 - j33.6\ \Omega$, $47.6 - j17.5\ \Omega$, (b) $\lambda/6$.

EXAMPLE 11.5

A $100 + j150$–Ω load is connected to a 75-Ω lossless line. Find:

(a) Γ
(b) s
(c) The load admittance Y_L
(d) Z_{in} at 0.4λ from the load
(e) The locations of V_{max} and V_{min} with respect to the load if the line is 0.6λ long
(f) Z_{in} at the generator.

Solution:

(a) We can use the Smith chart to solve this problem. The normalized load impedance is

$$z_L = \frac{Z_L}{Z_o} = \frac{100 + j150}{75} = 1.33 + j2$$

We locate this at point P on the Smith chart of Figure 11.16. At P, we obtain

$$|\Gamma| = \frac{OP}{OQ} = \frac{6 \text{ cm}}{9.1 \text{ cm}} = 0.659$$

$$\theta_\Gamma = \text{angle } POS = 40^\circ$$

Hence,

$$\Gamma = 0.659 \underline{/40^\circ}$$

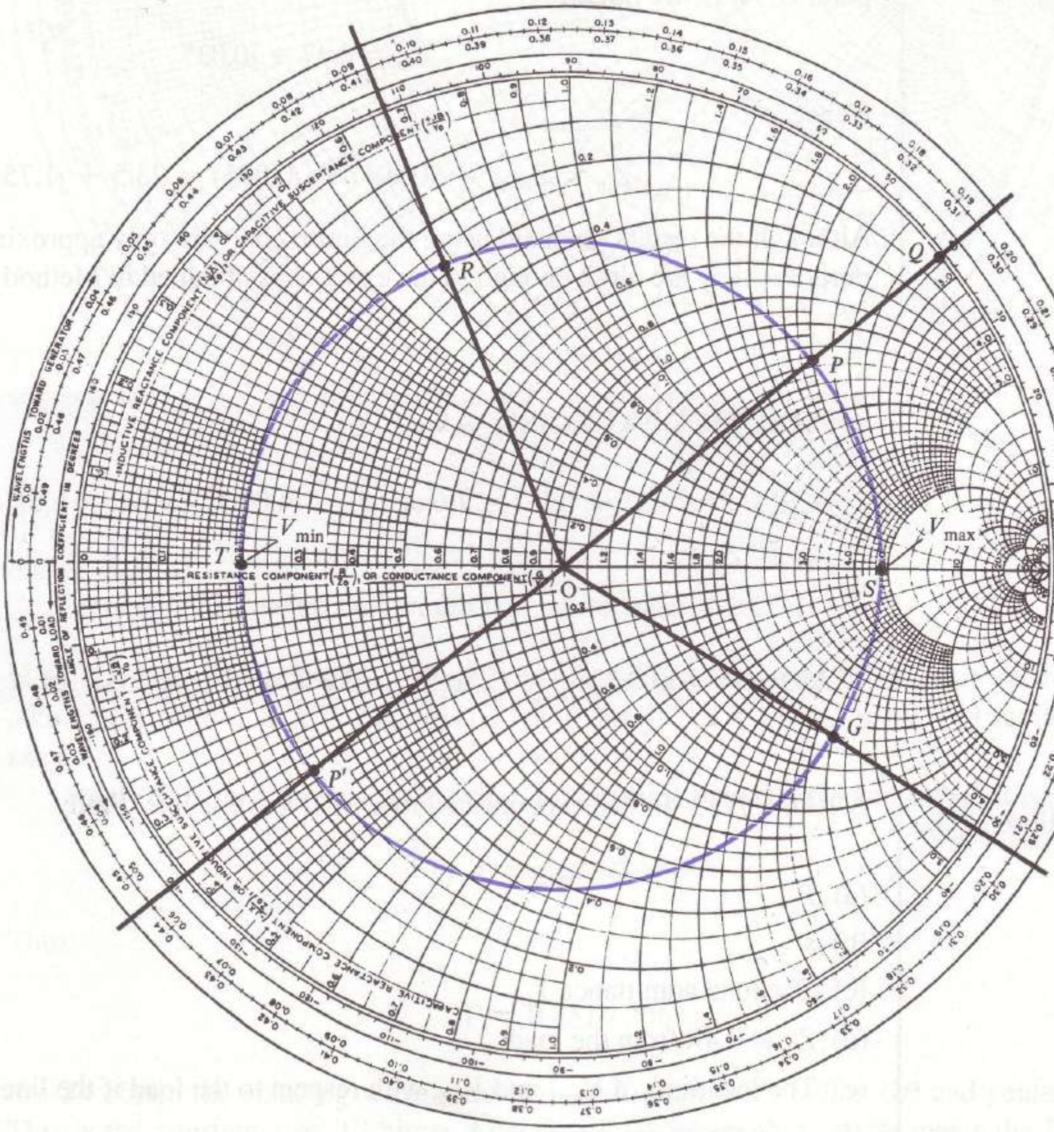


Figure 11.16 For Example 11.5.

Check:

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{100 + j150 - 75}{100 + j150 + 75}$$
$$= 0.659 \underline{/40^\circ}$$

(b) Draw the constant s-circle passing through P and obtain

$$s = 4.82$$

Check:

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.659}{1 - 0.659} = 4.865$$

(c) To obtain Y_L, extend PO to POP' and note point P' where the constant s-circle meets POP'. At P', obtain

$$y_L = 0.228 - j0.35$$

The load admittance is

$$Y_L = Y_o y_L = \frac{1}{75}(0.228 - j0.35) = 3.04 - j4.67 \text{ mS}$$

Check:

$$Y_L = \frac{1}{Z_L} = \frac{1}{100 + j150} = 3.07 - j4.62 \text{ mS}$$

(d) 0.4λ corresponds to an angular movement of $0.4 \times 720^\circ = 288^\circ$ on the constant s-circle. From P, we move 288° toward the generator (clockwise) on the s-circle to reach point R. At R,

$$z_{in} = 0.3 + j0.63$$

Hence

$$Z_{in} = Z_o z_{in} = 75\,(0.3 + j0.63)$$
$$= 22.5 + j47.25\ \Omega$$

Check:

$$\beta\ell = \frac{2\pi}{\lambda}(0.4\lambda) = 360^\circ\,(0.4) = 144^\circ$$

$$Z_{in} = Z_o\left[\frac{Z_L + jZ_o \tan \beta\ell}{Z_o + jZ_L \tan \beta\ell}\right]$$
$$= \frac{75\,(100 + j150 + j75 \tan 144^\circ)}{[75 + j(100 + j150) \tan 144^\circ]}$$
$$= 54.41\underline{/65.25^\circ}$$

or

$$Z_{in} = 21.9 + j47.6\ \Omega$$

(e) 0.6λ corresponds to an angular movement of

$$0.6 \times 720° = 432° = 1 \text{ revolution} + 72°$$

Thus, we start from P (load end), move along the s-circle 432°, or one revolution plus 72°, and reach the generator at point G. Note that to reach G from P, we have passed through point T (location of V_{min}) once and point S (location of V_{max}) twice. Thus, from the load,

$$1\text{st } V_{max} \text{ is located at } \frac{40°}{720°}\lambda = 0.055\lambda$$

$$2\text{nd } V_{max} \text{ is located at } 0.0555\lambda + \frac{\lambda}{2} = 0.555\lambda$$

and the only V_{min} is located at $0.055\lambda + \lambda/4 = 0.3055\lambda$

(f) At G (generator end),

$$z_{in} = 1.8 - j2.2$$

$$Z_{in} = 75(1.8 - j2.2) = 135 - j165\ \Omega.$$

This can be checked by using eq. (11.34), where $\beta\ell = \frac{2\pi}{\lambda}(0.6\lambda) = 216°$.

We can see how much time and effort is saved using the Smith chart.

PRACTICE EXERCISE 11.5

A lossless 60-Ω line is terminated by a $60 + j60$-Ω load.

(a) Find Γ and s. If $Z_{in} = 120 - j60\ \Omega$, how far (in terms of wavelengths) is the load from the generator? Solve this without using the Smith chart.

(b) Solve the problem in (a) using the Smith chart. Calculate Z_{max} and $Z_{in,min}$. How far (in terms of λ) is the first maximum voltage from the load?

Answer: (a) $0.4472\underline{/63.43°}$, 2.618, $\frac{\lambda}{8}(1 + 4n)$, $n = 0, 1, 2, \ldots$, (b) $0.4457\underline{/62°}$, 2.612, $\frac{\lambda}{8}(1 + 4n)$, 157.1 Ω, 22.92 Ω, 0.0861 λ.

Transmission Lines and Waveguides

Unit-3
Guided Waves

Contents

1.General field equations
2.Transverse Electric waves(TE Waves)
3. Transverse Magnetic waves(TM Waves)
4.Characteristics of TE, TM waves
5.TEM Wave
6.Velocities of wave propagation
7.Wave lengths
8.Wave impedances
9.Attenuation factor

Boundary conditions

Boundary conditions at the boundary between conductor and freespace

$$E_t = 0 \qquad \text{(1)}$$

$$D_n = \rho_s \Rightarrow E_n = \frac{\rho_s}{\varepsilon_0} \Rightarrow E_n \neq 0 \qquad \text{(2)}$$

$$H_t = K \Rightarrow H_t \neq 0 \qquad \text{(3)}$$

$$B_n = 0 \Rightarrow H_n = 0 \qquad \text{(4)}$$

Boundary conditions

(1)Electric field intensity which is tangential to conducting surface is equal to zero
(2)Electric flux density which is normal to conducting surface is equal to surface charge density on the conducting surface i.e Normal component of electric field intensity is not equal to zero
(3)Magnetic field intensity which is tangential to conducting surface is equal to surface current
(4) Magnetic field intensity which is normal to conducting surface is equal to zero

Basics of EM wave propagation

$$\nabla^2 E = \gamma^2 E$$

$$\nabla^2 H = \gamma^2 H$$

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$If\, \sigma = 0 \Rightarrow \gamma = j\omega\sqrt{\mu\varepsilon}$$

$$\nabla^2 E = -\omega^2 \mu\varepsilon E$$

$$\nabla^2 H = -\omega^2 \mu\varepsilon H$$

Basics of EM wave propagation

$$\bar{\gamma} = \bar{\alpha} + j\bar{\beta}$$

$$e^{j\omega t} e^{-\bar{\gamma} z} = e^{j\omega t} e^{-(\bar{\alpha} + j\bar{\beta}) z} = e^{-\bar{\alpha} z} e^{j(\omega t - \bar{\beta} z)}$$

$$H = H_0 e^{-\bar{\gamma} z}$$

$$\frac{\partial H}{\partial z} = \frac{\partial H_0 e^{-\bar{\gamma} z}}{\partial z} = -\bar{\gamma} H_0 e^{-\bar{\gamma} z} = -\bar{\gamma} H$$

Basics of EM wave propagation

$$\frac{\partial H}{\partial z} = \frac{\partial H_0 e^{-\bar{\gamma} z}}{\partial z} = -\bar{\gamma} H_0 e^{-\bar{\gamma} z} = -\bar{\gamma} H$$

$$\frac{\partial}{\partial z} = -\bar{\gamma}$$

Parallel Plane Waveguide

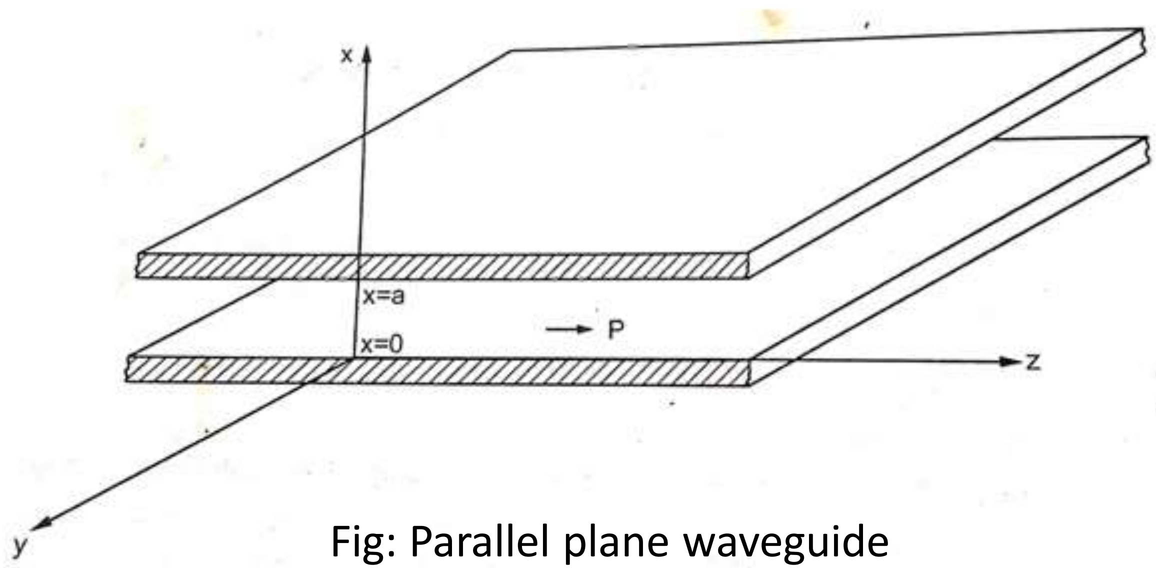


Fig: Parallel plane waveguide

Along X-direction boundaries are defined i.e $\frac{\partial}{\partial x} \neq 0$

Along Y-direction Plates are infinitely extended i.e $\frac{\partial}{\partial y} = 0$

Along Z-direction EM Wave is travelling i .e $\frac{\partial}{\partial z} = -\overline{\gamma}$

General field Equations

$$\nabla \times H = j\omega\varepsilon E \qquad (1)$$

$$\nabla \times E = -j\omega\mu H \qquad (2)$$

$$\begin{bmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{bmatrix} = j\omega\varepsilon(E_x a_x + E_y a_y + E_z a_z) \qquad (3)$$

$$\begin{bmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{bmatrix} = -j\omega\mu(H_x a_x + H_y a_y + H_z a_z) \qquad (4)$$

General field Equations

From (3)

$$\frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y = j\omega\varepsilon E_x \quad \text{——— (5)}$$

$$\frac{\partial}{\partial z}H_x - \frac{\partial}{\partial x}H_z = j\omega\varepsilon E_y \quad \text{——— (6)}$$

$$\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x = j\omega\varepsilon E_z \quad \text{——— (7)}$$

From (5), (6), (7)

$$\bar{\gamma}H_y = j\omega\varepsilon E_x \quad \text{——— (8)}$$

$$-\bar{\gamma}H_x - \frac{\partial}{\partial x}H_z = j\omega\varepsilon E_y \quad \text{——— (9)}$$

$$\frac{\partial}{\partial x}H_y = j\omega\varepsilon E_z \quad \text{——— (10)}$$

General field Equations

From (4)

$$-\frac{\partial}{\partial z}E_y + \frac{\partial}{\partial y}E_z = -j\omega\mu H_x \quad \text{——————} (11)$$

$$-\frac{\partial}{\partial x}E_z + \frac{\partial}{\partial z}E_x = -j\omega\mu H_y \quad \text{——————} (12)$$

$$-\frac{\partial}{\partial y}E_x + \frac{\partial}{\partial x}E_y = -j\omega\mu H_z \quad \text{——————} (13)$$

From (11), (12), (13)

$$\bar{\gamma} E_y = -j\omega\mu H_x \quad \text{——————} (14)$$

$$-\frac{\partial}{\partial x}E_z - \bar{\gamma} E_x = -j\omega\mu H_y \quad \text{——————} (15)$$

$$\frac{\partial}{\partial x}E_y = -j\omega\mu H_z \quad \text{——————} (16)$$

General field Equations

From (8) $$H_y = \frac{j\omega\varepsilon}{\bar{\gamma}} E_x \qquad \text{———— (17)}$$

From (15),(17) $$-\frac{\partial}{\partial x} E_z - \bar{\gamma} E_x = \frac{\omega^2 \mu\varepsilon}{\bar{\gamma}} E_x$$

$$-\frac{\partial}{\partial x} E_z = (\bar{\gamma} + \frac{\omega^2 \mu\varepsilon}{\bar{\gamma}}) E_x$$

$$E_x = \frac{-\bar{\gamma}}{\bar{\gamma}^2 + \omega^2 \mu\varepsilon} \frac{\partial E_Z}{\partial x}$$

Let $$\bar{\gamma}^2 + \omega^2 \mu\varepsilon = h^2$$

General field Equations

$$\left\{\begin{aligned} E_x &= \frac{-\bar{\gamma}}{h^2}\frac{\partial E_Z}{\partial x} \\ E_y &= \frac{j\omega\mu}{h^2}\frac{\partial H_Z}{\partial x} \\ H_x &= \frac{-\bar{\gamma}}{h^2}\frac{\partial H_Z}{\partial x} \\ H_y &= \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_Z}{\partial x} \end{aligned}\right\}$$

TE Waves

$$E_z = 0$$

$$H_z, E_y, H_x \neq 0$$

$$\nabla^2 E = -\omega^2 \mu\varepsilon E$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} = -\omega^2 \mu\varepsilon E_y$$

$$\frac{\partial^2 E_y}{\partial x^2} + \bar{\gamma}^2 E_y = -\omega^2 \mu\varepsilon E_y$$

$$\frac{\partial^2 E_y}{\partial x^2} = -(\bar{\gamma}^2 + \omega^2 \mu\varepsilon) E_y$$

Let $\bar{\gamma}^2 + \omega^2 \mu\varepsilon = h^2$

TE Waves

$$\frac{\partial^2 E_y}{\partial x^2} = -h^2 E_y$$

Solution of above 2nd order differential equation is

$$E_y = c_1 \sinh x + c_2 \cosh x \qquad \text{———— (I)}$$

c_1, c_2 are constants

At x=0 $\quad E_y = 0$

Apply above boundary condition in I

$$c_2 = 0$$

$$E_y = c_1 \sinh x \qquad \text{———— (II)}$$

At x=a $\quad E_y = 0$

Apply above boundary condition in II

$$c_1 \, \mathrm{sinha} = 0$$

$$\mathrm{ha} = \mathrm{m}\pi$$; m=0,1,2,3..

TE Waves

$$\mathrm{h} = \frac{m\pi}{a}$$

$$E_y = c_1 \sin \frac{m\pi}{a} x$$

From (14) $$H_x = \frac{-\bar{\gamma}}{j\omega\mu} E_y$$

$$H_x = \frac{-\bar{\gamma}}{j\omega\mu} c_1 \sin \frac{m\pi}{a} x$$

From (16) $$\frac{\partial E_y}{\partial x} = -j\omega\mu H_z$$

$$H_z = \frac{-1}{j\omega\mu} \frac{\partial E_y}{\partial x}$$

TE Waves

$$H_z = \frac{-m\pi c_1}{j\omega\mu a}\cos\frac{m\pi}{a}x$$

$$E_y = c_1 \sin(\frac{m\pi}{a}x)e^{-\bar{\gamma}z}$$

$$H_x = \frac{-\bar{\gamma}}{j\omega\mu}c_1 \sin(\frac{m\pi}{a}x)e^{-\bar{\gamma}z}$$

$$H_z = \frac{-m\pi c_1}{j\omega\mu a}\cos(\frac{m\pi}{a}x)e^{-\bar{\gamma}z}$$

$$\left\{\begin{array}{l} E_y = c_1 \sin(\frac{m\pi}{a}x)e^{-j\bar{\beta}z} \\ H_x = \frac{-\bar{\beta}}{\omega\mu}c_1 \sin(\frac{m\pi}{a}x)e^{-j\bar{\beta}z} \\ H_z = \frac{-m\pi c_1}{j\omega\mu a}\cos(\frac{m\pi}{a}x)e^{-j\bar{\beta}z} \end{array}\right\}$$

For different values of integer m there are different field configurations and these field configurations are called as modes(TE_{m0})

If m=0 all components are equal to zero.i.e. TE_{00} Mode does not possible

If $m=1 \Rightarrow E_y, H_x \neq 0, \mathrm{H}_z \neq 0$

i.e TE_{10} is Least possible mode in TE_{m0} waves

TM Waves

$$H_z = 0$$

$$E_z, E_x, H_y \neq 0$$

$$\nabla^2 H = -\omega^2 \mu\varepsilon H$$

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} = -\omega^2 \mu\varepsilon H_y$$

$$\frac{\partial^2 H_y}{\partial x^2} + \overline{\gamma}^2 H_y = -\omega^2 \mu\varepsilon H_y$$

$$\frac{\partial^2 H_y}{\partial x^2} = -(\overline{\gamma}^2 + \omega^2 \mu\varepsilon)\mathrm{H}_y$$

$$\frac{\partial^2 H_y}{\partial x^2} = -h^2 H_y$$

TM Waves

$$H_y = c_3 \sinh x + c_4 \cosh x$$

$$\frac{\partial}{\partial x} H_y = j\omega\varepsilon E_z$$

$$E_z = \frac{1}{j\omega\varepsilon}(c_3 h \cosh x - c_4 h \sinh x)$$

$$E_z = \frac{h}{j\omega\varepsilon}(c_3 \cosh x - c_4 \sinh x)$$

At x=0 ; $E_z = 0$

$$c_3 = 0$$

$$E_z = \frac{-hc_4 \sinh x}{j\omega\varepsilon}$$

TM Waves

At x=a ; $E_z = 0$

$$\sinh a = 0$$

$$ha = m\pi \qquad \text{m=0,1,2,3..}$$

$$h = \frac{m\pi}{a}$$

$$E_z = \frac{-hc_4}{j\omega\varepsilon}\sin(\frac{m\pi}{a}x)$$

$$\frac{\partial}{\partial x}H_y = j\omega\varepsilon E_z \Longrightarrow H_y = c_4\cos(\frac{m\pi}{a}x)$$

$$\bar{\gamma}H_y = j\omega\varepsilon E_x \Longrightarrow E_x = \frac{\bar{\gamma}c_4}{j\omega\varepsilon}\cos(\frac{m\pi}{a}x)$$

TM Waves

$$E_x = \frac{\bar{\gamma} c_4}{j\omega\varepsilon} \cos(\frac{m\pi}{a} x) e^{-\bar{\gamma} z}$$

$$H_y = c_4 \cos(\frac{m\pi}{a} x) e^{-\bar{\gamma} z}$$

$$E_z = \frac{-m\pi c_4}{j\omega\varepsilon a} \sin(\frac{m\pi}{a} x) e^{-\bar{\gamma} z}$$

$$\left[\begin{aligned} E_x &= \frac{\bar{\beta} c_4}{\omega\varepsilon} \cos(\frac{m\pi}{a} x) e^{-j\bar{\beta} z} \\ H_y &= c_4 \cos(\frac{m\pi}{a} x) e^{-j\bar{\beta} z} \\ E_z &= \frac{-m\pi c_4}{j\omega\varepsilon a} \sin(\frac{m\pi}{a} x) e^{-j\bar{\beta} z} \end{aligned} \right]$$

For different values of integer m there are different field configurations and these field configurations are called as modes(TM_{mo})

If $m = 0 \Rightarrow E_x, H_y \neq 0, E_z = 0$

TM_{00} Mode is possible and it is Least possible mode In TM_{mo} waves

Characteristics of TE, TM waves

Cutoff frequency

$$\bar{\gamma}^2 + \omega^2 \mu \varepsilon = h^2$$

$$h = \frac{m\pi}{a}$$

$$\bar{\gamma}^2 = (\frac{m\pi}{a})^2 - \omega^2 \mu \varepsilon$$

$$\bar{\gamma} = \sqrt{(\frac{m\pi}{a})^2 - \omega^2 \mu \varepsilon}$$

If $\omega^2 \mu \varepsilon > (\frac{m\pi}{a})^2$

$$\bar{\gamma} = j\bar{\beta} = j\sqrt{\omega^2 \mu \varepsilon - (\frac{m\pi}{a})^2}$$

Characteristics of TE, TM waves

$$\bar{\beta} = \sqrt{\omega^2 \mu\varepsilon - (\frac{m\pi}{a})^2}$$

$$At\omega = \omega_c \Rightarrow \bar{\beta} = 0$$

$$\omega_c{}^2 \mu\varepsilon = (\frac{m\pi}{a})^2$$

$$\omega_c \sqrt{\mu\varepsilon} = \frac{m\pi}{a}$$

$$f_c = \frac{m\pi}{2\pi a\sqrt{\mu\varepsilon}}$$

$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}}$$

Characteristics of TE, TM waves

If the medium between two conducting plates is freespace

$$f_c = \frac{mc}{2a}$$

If $\omega < \omega_c (f < f_c)$

$\bar{\gamma}$ Is purely real and there is no wave propagation

If $\omega > \omega_c (f > f_c)$

$\bar{\gamma}$ Is purely imaginary and there is wave propagation

To get wave propagation between two conducting walls operating frequency f must be greater than cutoff frequency and we can say parallel plane waveguide Acting as High Pass Filter

Characteristics of TE, TM waves

Cutoff frequency is defined as the frequency below which wave propagation ceases

$$f_c = \frac{mc}{2a}$$

$$\lambda_c = \frac{c}{f_c}$$

Cutoff wave length $\lambda_c = \frac{2a}{m}$

The mode which is having least cutoff frequency and highest cutoff wave length is called as dominant mode

$$f_{cTE_{10}} = \frac{c}{2a}$$

$$\lambda_{cTE_{10}} = 2a$$

Characteristics of TE, TM waves

$$f_{cTM_{00}} = 0$$

In TE_{m0} waves dominant mode is TE_{10}

In TM_{m0} waves dominant mode is TM_{00}

TM_{00} is dominant mode for the entire parallel plane waveguide

TEM Waves

TM wave ($H_z = 0$) equations are

$$E_x = \frac{\bar{\beta} c_4}{\omega\varepsilon} \cos(\frac{m\pi}{a} x) e^{-j\bar{\beta} z}$$

$$H_y = c_4 \cos(\frac{m\pi}{a} x) e^{-j\bar{\beta} z}$$

$$E_z = \frac{-m\pi c_4}{j\omega\varepsilon a} \sin(\frac{m\pi}{a} x) e^{-j\bar{\beta} z}$$

For m=0

$$E_x = \frac{\bar{\beta} c_4}{\omega\varepsilon} e^{-j\bar{\beta} z}$$

$$H_y = c_4 e^{-j\bar{\beta} z}$$

$$E_z = 0$$

TEM Waves

$E_z = 0 \quad , \quad H_z = 0 \longrightarrow$ TEM wave is possible between two conducting walls

TEM wave is also called as principal wave

Properties of TEM wave

1.Fields are entirely transverse

2.The amplitudes of field components along the normal to wave propagation are constant

$$\bar{\beta} = \sqrt{\omega^2 \mu\varepsilon - (\frac{m\pi}{a})^2}$$

$$\bar{\beta} = \omega\sqrt{\mu\varepsilon}$$

$$\frac{\bar{\beta} c_4}{\omega\varepsilon} = \frac{\omega\sqrt{\mu\varepsilon}}{\omega\varepsilon} c_4 = \sqrt{\frac{\mu}{\varepsilon}} c_4$$

TEM Waves

3.Cutoff frequency is zero indicating that all the frequencies down to zero can propagate along the guide

$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}}$$

For m=0 $f_c = 0$

4.Velocity of TEM wave is equal to velocity of light if the medium between conducting walls is freespace

$$\bar{v} = \frac{\omega}{\bar{\beta}} = \frac{\omega}{\omega\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

5.The ratio between E-field and magnetic field components is called as intrinsic impedance given by

$$\eta = \frac{E_x}{H_y} = \frac{\bar{\beta}c_4}{\omega\varepsilon c_4} = \frac{\bar{\beta}}{\omega\varepsilon} = \frac{\omega\sqrt{\mu\varepsilon}}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}$$

For freespace

$$\eta = 120\pi$$

Velocities of wave propagation

$$\bar{\beta} = \sqrt{\omega^2 \mu\varepsilon - (\frac{m\pi}{a})^2}$$

$$At \omega = \omega_c \Rightarrow \bar{\beta} = 0$$

$$\omega_c{}^2 \mu\varepsilon = (\frac{m\pi}{a})^2$$

$$\bar{\beta} = \sqrt{\omega^2 \mu\varepsilon - \omega_c{}^2 \mu\varepsilon}$$

Velocity of single frequency component along the guide is called as phase velocity

$$\bar{v} = \frac{\omega}{\bar{\beta}}$$

Velocities of wave propagation

$$\bar{v} = \frac{\omega}{\sqrt{\omega^2 \mu\varepsilon - \omega_c{}^2 \mu\varepsilon}}$$

$$\bar{v} = \frac{\omega}{\sqrt{\omega^2 \mu\varepsilon(1 - \frac{\omega_c{}^2 \mu\varepsilon}{\omega^2 \mu\varepsilon})}}$$

$$\bar{v} = \frac{1 / \sqrt{\mu\varepsilon}}{\sqrt{(1 - \frac{f_c{}^2}{f^2})}}$$

For freespace $$\bar{v} = \frac{c}{\sqrt{(1 - \frac{f_c{}^2}{f^2})}}$$

Velocities of wave propagation

Velocity of resultant energy propagation along the guide is called as group velocity

$$\bar{\beta} = \sqrt{\omega^2 \mu\varepsilon - \omega_c{}^2 \mu\varepsilon}$$

$$v_g = \left(\frac{d\bar{\beta}}{d\omega}\right)^{-1}$$

$$\frac{d\bar{\beta}}{d\omega} = \frac{1}{2\sqrt{\omega^2 \mu\varepsilon - \omega_c{}^2 \mu\varepsilon}} 2\omega\mu\varepsilon$$

$$\frac{d\bar{\beta}}{d\omega} = \frac{\omega\mu\varepsilon}{\sqrt{\omega^2 \mu\varepsilon\left(1 - \frac{\omega_c{}^2 \mu\varepsilon}{\omega^2 \mu\varepsilon}\right)}}$$

Velocities of wave propagation

$$\frac{d\bar{\beta}}{d\omega} = \frac{1}{c\sqrt{(1-\frac{f_c^{\,2}}{f^{\,2}})}}$$

$$v_g = c\sqrt{(1-\frac{f_c^{\,2}}{f^{\,2}})}$$

$$\bar{v}v_g = \frac{c}{\sqrt{(1-\frac{f_c^{\,2}}{f^{\,2}})}}c\sqrt{(1-\frac{f_c^{\,2}}{f^{\,2}})}$$

$$\bar{v}v_g = c^2$$

$$\bar{v} > c > v_g$$

Wavelengths

Relation between cutoff wavelength(λ_c), guided wave length($\bar{\lambda}$) and freespace wavelength(λ)

$$\bar{\beta} = \sqrt{\omega^2 \mu\varepsilon - \omega_c^{\ 2} \mu\varepsilon}$$

$$\frac{2\pi}{\bar{\lambda}} = \omega\sqrt{\mu\varepsilon}\sqrt{(1 - \frac{f_c^{\ 2}}{f^{\ 2}})}$$

$$\frac{2\pi}{\bar{\lambda}} = \frac{\omega}{c}\sqrt{(1 - \frac{f_c^{\ 2}}{f^{\ 2}})}$$

$$\frac{2\pi}{\bar{\lambda}} = \frac{2\pi f}{c}\sqrt{(1 - \frac{f_c^{\ 2}}{f^{\ 2}})}$$

$$\frac{2\pi}{\bar{\lambda}} = \frac{2\pi f}{f\lambda}\sqrt{(1 - \frac{\lambda^2}{\lambda_c^{\ 2}})}$$

$$c = f\lambda$$

$$c = f_c\lambda_c$$

Wavelengths

$$\frac{1}{\bar{\lambda}} = \frac{1}{\lambda}\sqrt{(1-\frac{\lambda^2}{\lambda_c{}^2})}$$

$$\frac{1}{\bar{\lambda}^2} = \frac{1}{\lambda^2} - \frac{1}{\lambda_c{}^2}$$

$$\frac{1}{\lambda^2} = \frac{1}{\bar{\lambda}^2} + \frac{1}{\lambda_c{}^2}$$

Wave impedances

$$E_x = \frac{-\bar{\gamma}}{h^2}\frac{\partial E_Z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_Z}{\partial x}$$

$$H_y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_Z}{\partial x}$$

$$H_x = \frac{-\bar{\gamma}}{h^2}\frac{\partial H_Z}{\partial x}$$

TM wave

$$Z_{TM} = \frac{E_x}{H_y} = \frac{\bar{\gamma}}{j\omega\varepsilon}$$

$$Z_{TM} = \frac{j\bar{\beta}}{j\omega\varepsilon}$$

$$Z_{TM} = \frac{\bar{\beta}}{\omega\varepsilon}$$

Wave impedances

$$\bar{\beta} = \sqrt{\omega^2 \mu\varepsilon - \omega_c{}^2 \mu\varepsilon}$$

$$Z_{TM} = \frac{\sqrt{\omega^2 \mu\varepsilon - \omega_c{}^2 \mu\varepsilon}}{\omega\varepsilon}$$

$$Z_{TM} = \frac{\sqrt{\omega^2 \mu\varepsilon (1 - \frac{\omega_c{}^2 \mu\varepsilon}{\omega^2 \mu\varepsilon})}}{\omega\varepsilon}$$

$$Z_{TM} = \sqrt{\frac{\mu}{\varepsilon}} \sqrt{1 - \frac{f_c{}^2}{f^2}}$$

$$Z_{TM} = \eta \sqrt{1 - \frac{f_c{}^2}{f^2}}$$

Wave impedances

TE wave

$$Z_{TE} = \frac{-E_y}{H_x} = \frac{j\omega\mu}{\bar{\gamma}}$$

$$Z_{TE} = \frac{j\omega\mu}{j\bar{\beta}}$$

$$Z_{TE} = \frac{\omega\mu}{\bar{\beta}}$$

$$Z_{TE} = \frac{\omega\mu}{\sqrt{\omega^2\mu\varepsilon - \omega_c{}^2\mu\varepsilon}}$$

$$Z_{TE} = \frac{\omega\mu}{\sqrt{\omega^2\mu\varepsilon(1 - \frac{\omega_c{}^2\mu\varepsilon}{\omega^2\mu\varepsilon})}}$$

Wave impedances

TE wave

$$Z_{TE} = \sqrt{\frac{\mu}{\varepsilon}} / \sqrt{1 - \frac{f_c^{\,2}}{f^{\,2}}}$$

$$Z_{TE} = \frac{\eta}{\sqrt{1 - \frac{f_c^{\,2}}{f^{\,2}}}}$$

$$Z_{TE} Z_{TM} = \frac{\eta}{\sqrt{1 - \frac{f_c^{\,2}}{f^{\,2}}}} \eta \sqrt{1 - \frac{f_c^{\,2}}{f^{\,2}}}$$

$$Z_{TE} Z_{TM} = \eta^2$$

$$Z_{TE} > \eta > Z_{TM}$$

Attenuation factor

$$\alpha = \frac{powerloss / unitlength}{2 \times averagepowertransmitted}$$

TEM Wave

$$E_x = \frac{\bar{\beta} c_4}{\omega \varepsilon} e^{-j\bar{\beta} z}$$

$$H_y = c_4 e^{-j\bar{\beta} z}$$

$$E_z = 0$$

Surface current $J_s = |H_{\tan}| = \mathrm{c}_4$

Power loss in each conducting wall = $\frac{1}{2} J_s^{\,2} R_s = \frac{1}{2} c_4^{\,2} R_s$

Power loss due to two conducting walls = $c_4^{\,2} R_s$

Attenuation factor

TEM Wave

Power loss per unit length for width b of conducting plates = $c_4^{\,2} R_s b$

Average power transmitted along guide per unit area = $P_{avg} = \frac{1}{2}\mathrm{Re}(E \times H^*)$

$$P_{avg} = \frac{1}{2}\mathrm{Re}\left(\frac{\bar{\beta} c_4}{\omega\varepsilon} e^{-j\bar{\beta} z} \times c_4 e^{j\bar{\beta} z}\right)$$

$$P_{avg} = \frac{\bar{\beta} c_4^{\,2}}{2\omega\varepsilon}$$

$$P_{avg} = \frac{1}{2}\eta c_4^{\,2} ab$$

Attenuation factor

TEM Wave

$$\alpha = \frac{powerloss / unitlength}{2 \times averagepowertransmitted}$$

$$\alpha_{TEM} = \frac{{c_4}^2 R_s b}{2 \times \frac{1}{2} \eta {c_4}^2 ab}$$

$$\alpha_{TEM} = \frac{R_s}{\eta a}$$

$$\alpha_{TEM} = \frac{1}{\eta a} \sqrt{\frac{\omega \mu}{2 \sigma}}$$

$$\alpha_{TEM} \propto \sqrt{f}$$

Attenuation factor

TE Wave

$$E_y = c_1 \sin(\frac{m\pi}{a} x) e^{-j\bar{\beta} z}$$

$$H_x = \frac{-\bar{\beta}}{\omega\mu} c_1 \sin(\frac{m\pi}{a} x) e^{-j\bar{\beta} z}$$

$$H_z = \frac{-m\pi c_1}{j\omega\mu a} \cos(\frac{m\pi}{a} x) e^{-j\bar{\beta} z}$$

Surface current = $J_s = |H_{\tan}| = \frac{m\pi c_1}{\omega\mu a}$

Power loss due to two conducting walls = $J_s{}^2 R_s = (\frac{m\pi c_1}{\omega\mu a})^2 R_s$

Average power transmitted along guide per unit area = $P_{avg} = \frac{1}{2} \mathrm{Re}(E \times H^*)$

Attenuation factor

TE Wave

$$E \times H^* = \begin{bmatrix} a_x & a_y & a_z \\ 0 & E_y & 0 \\ H_x & 0 & H_z \end{bmatrix} = -E_y H_x^*$$

$$P_{avg} = \frac{1}{2} \operatorname{Re}(-E_y H_x^*)$$

$$P_{avg} = \frac{\bar{\beta} c_1^2}{2\omega\mu} \sin^2(\frac{m\pi}{a} x)$$

Total power transmitted along guide for spacing ‘a’ between plates = $\int_0^a \frac{\bar{\beta} c_1^2}{2\omega\mu} \sin^2(\frac{m\pi}{a} x) dx$

$$= \frac{\bar{\beta} c_1^2}{2\omega\mu} (\frac{a}{2})$$

Attenuation factor

TE Wave

$$\alpha = \frac{powerloss / unitlength}{2 \times averagepowertransmitted}$$

$$\alpha = \frac{(\frac{m\pi c_1}{\omega\mu a})^2 R_s}{2 \times \frac{\bar{\beta} c_1^{\ 2}}{2\omega\mu}(\frac{a}{2})}$$

$$\alpha = k \frac{\sqrt{\omega}}{\omega\sqrt{\omega^2 \mu\varepsilon - \omega_c^{\ 2} \mu\varepsilon}}$$

$$\alpha_{TE} = k \frac{1}{\omega^{3/2}\sqrt{1 - \frac{f_c^{\ 2}}{f^2}}}$$

If $f >> f_c \Rightarrow \alpha_{TE} \propto \frac{1}{f^{3/2}}$

Attenuation factor

TM Wave

$$E_x = \frac{\bar{\beta} c_4}{\omega\varepsilon} \cos(\frac{m\pi}{a} x) e^{-j\bar{\beta} z}$$

$$H_y = c_4 \cos(\frac{m\pi}{a} x) e^{-j\bar{\beta} z}$$

$$E_z = \frac{-m\pi c_4}{j\omega\varepsilon a} \sin(\frac{m\pi}{a} x) e^{-j\bar{\beta} z}$$

Surface current $J_s = |H_{\tan}| = \mathrm{c}_4$

Power loss due to two conducting walls = $J_s{}^2 R_s = c_4{}^2 R_s$

Average power transmitted along guide per unit area = $P_{avg} = \frac{1}{2} \mathrm{Re}(E \times H^*)$

$$P_{avg} = \frac{1}{2} \mathrm{Re}(E_x H_y *)$$

Attenuation factor

TM Wave

$$P_{avg} = \frac{\bar{\beta} c_4{}^2}{2\omega\varepsilon} \cos^2(\frac{m\pi}{a} x)$$

Total power transmitted along guide for spacing ‘a’ between plates = $\int_0^a \frac{\bar{\beta} c_4{}^2}{2\omega\varepsilon} \cos^2(\frac{m\pi}{a} x) dx$

$$= \frac{\bar{\beta} c_4{}^2}{2\omega\varepsilon} (\frac{a}{2})$$

$$\alpha = \frac{powerloss / unitlength}{2 \times averagepowertransmitted}$$

$$\alpha = \frac{c_4{}^2 R_s}{2 \times \frac{\bar{\beta} c_4{}^2}{2\omega\varepsilon} (\frac{a}{2})}$$

Attenuation factor

TM Wave

$$\alpha = k \frac{\omega\sqrt{\omega}}{\sqrt{\omega^2 \mu\varepsilon - \omega_c{}^2 \mu\varepsilon}}$$

$$\alpha = k \frac{\omega\sqrt{\omega}}{\omega\sqrt{1 - \frac{f_c{}^2}{f^2}}}$$

$$\alpha_{TM} = k \frac{\sqrt{\omega}}{\sqrt{1 - \frac{f_c{}^2}{f^2}}}$$

If $f >> f_c \Rightarrow \alpha_{TM} \propto \sqrt{f}$

Attenuation factor

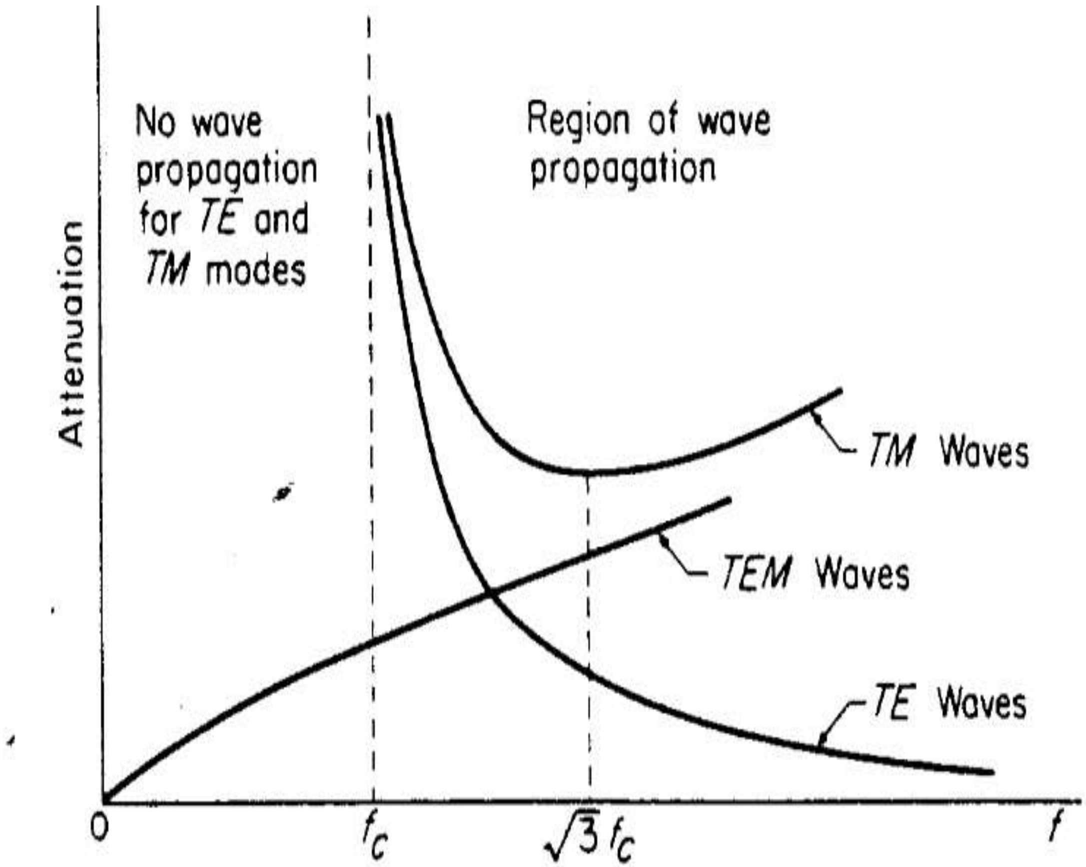


Figure Attenuation-versus-frequency characteristics of waves guided between parallel conducting plates.

Unit-3
Rectangular wave guide

Contents

1.General field equations

2.Impossibility of TEM wave

3.Transverse Magnetic waves(TM Waves)

4. Transverse Electric waves(TE Waves)

5.Cutoff frequency

General field Equations

$$\nabla \times H = j\omega\varepsilon E \qquad (1)$$

$$\nabla \times E = -j\omega\mu H \qquad (2)$$

$$\begin{bmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{bmatrix} = j\omega\varepsilon(E_x a_x + E_y a_y + E_z a_z) \qquad (3)$$

$$\begin{bmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{bmatrix} = -j\omega\mu(H_x a_x + H_y a_y + H_z a_z) \qquad (4)$$

General field Equations

From (3)

$$\frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y = j\omega\varepsilon E_x \quad \text{———} \quad (5)$$

$$\frac{\partial}{\partial z}H_x - \frac{\partial}{\partial x}H_z = j\omega\varepsilon E_y \quad \text{———} \quad (6)$$

$$\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x = j\omega\varepsilon E_z \quad \text{———} \quad (7)$$

From (5), (6), (7)

$$\frac{\partial}{\partial y}H_z + \overline{\gamma}H_y = j\omega\varepsilon E_x \quad \text{———} \quad (8)$$

$$-\overline{\gamma}H_x - \frac{\partial}{\partial x}H_z = j\omega\varepsilon E_y \quad \text{———} \quad (9)$$

$$\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x = j\omega\varepsilon E_z \quad \text{———} \quad (10)$$

General field Equations

From (4)

$$-\frac{\partial}{\partial z}E_y + \frac{\partial}{\partial y}E_z = -j\omega\mu H_x \quad \text{———— (11)}$$

$$-\frac{\partial}{\partial x}E_z + \frac{\partial}{\partial z}E_x = -j\omega\mu H_y \quad \text{———— (12)}$$

$$-\frac{\partial}{\partial y}E_x + \frac{\partial}{\partial x}E_y = -j\omega\mu H_z \quad \text{———— (13)}$$

From (11), (12), (13)

$$\bar{\gamma}E_y + \frac{\partial}{\partial y}E_z = -j\omega\mu H_x \quad \text{———— (14)}$$

$$-\frac{\partial}{\partial x}E_z - \bar{\gamma}E_x = -j\omega\mu H_y \quad \text{———— (15)}$$

$$-\frac{\partial}{\partial y}E_x + \frac{\partial}{\partial x}E_y = -j\omega\mu H_z \quad \text{———— (16)}$$

General field Equations

From (15) $$H_y = \frac{\bar{\gamma}}{j\omega\mu} E_x + \frac{1}{j\omega\mu}\frac{\partial}{\partial x} E_z \qquad \text{(17)}$$

Substitute (17) in (8)

$$\frac{\partial H_z}{\partial y} + \frac{\bar{\gamma}^2}{j\omega\mu} E_x + \frac{\bar{\gamma}}{j\omega\mu}\frac{\partial}{\partial x} E_z = j\omega\varepsilon E_x$$

$$j\omega\mu \frac{\partial H_z}{\partial y} + \bar{\gamma}^2 E_x + \bar{\gamma}\frac{\partial}{\partial x} E_z = -\omega^2 \mu\varepsilon E_x$$

$$j\omega\mu \frac{\partial H_z}{\partial y} + \bar{\gamma}\frac{\partial}{\partial x} E_z = -(\bar{\gamma}^2 + \omega^2 \mu\varepsilon) E_x$$

$$Let \bar{\gamma}^2 + \omega^2 \mu\varepsilon = h^2$$

General field Equations

$$\begin{cases} E_x = \dfrac{-j\omega\mu}{h^2}\dfrac{\partial H_z}{\partial y} - \dfrac{\bar{\gamma}}{h^2}\dfrac{\partial E_z}{\partial x} \\ E_y = \dfrac{j\omega\mu}{h^2}\dfrac{\partial H_z}{\partial x} - \dfrac{\bar{\gamma}}{h^2}\dfrac{\partial E_z}{\partial y} \\ H_x = \dfrac{-\bar{\gamma}}{h^2}\dfrac{\partial H_z}{\partial x} + \dfrac{j\omega\varepsilon}{h^2}\dfrac{\partial E_z}{\partial y} \\ H_y = \dfrac{-j\omega\varepsilon}{h^2}\dfrac{\partial E_z}{\partial x} - \dfrac{\bar{\gamma}}{h^2}\dfrac{\partial H_z}{\partial y} \end{cases}$$

Impossibility of TEM wave in a hallow waveguide

For TEM wave $E_z = H_z = 0$

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} - \frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} - \frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial y}$$

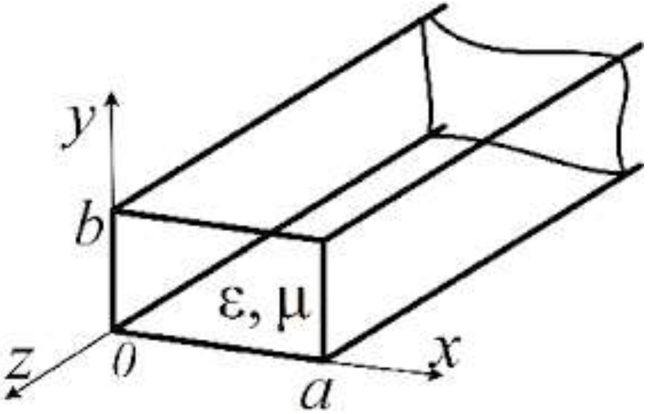
$$H_x = \frac{-\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial x} + \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y}$$

$$H_y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} - \frac{\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial y}$$

If $E_z = H_z = 0 \Rightarrow E_x = E_y = H_x = H_y = 0$

i.e All components are zeros means TEM wave is not possible in any hallow waveguide

TM wave



$$H_z = 0$$

$$\nabla^2 E = -\omega^2 \mu\varepsilon E$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu\varepsilon E_z$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = -(\bar{\gamma}^2 + \omega^2 \mu\varepsilon) E_z$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = -h^2 E_z$$

Let $E_z = X(x)Y(y)$

TM wave

$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} = -h^2 XY$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = -h^2$$

Let $\frac{1}{X}\frac{d^2 X}{dx^2} = -B^2$ ——— (I) ; $\frac{1}{Y}\frac{d^2 Y}{dy^2} = -A^2$ ——— (II)

$$B^2 + A^2 = h^2$$

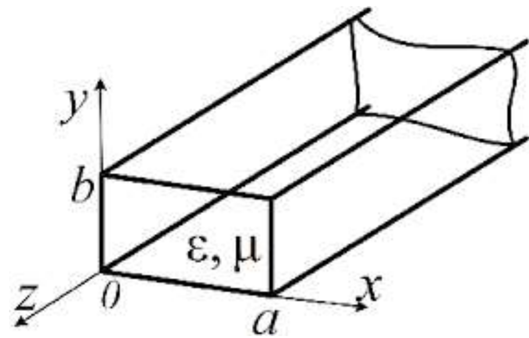
Solution of I and II are $X = c_1 \cos Bx + c_2 \sin Bx$

$$Y = c_3 \cos \mathrm{Ay} + c_4 \sin \mathrm{Ay}$$

TM wave

$$E_z = XY = (c_1 \cos Bx + c_2 \sin Bx)(c_3 \cos Ay + c_4 \sin Ay) \quad \text{—— (III)}$$

(i)Bottom conducting wall



$E_z = 0$ At y=0 $\forall x \rightarrow$ 0 to a

Apply above boundary condition in III

$$0 = (c_1 \cos Bx + c_2 \sin Bx)c_3$$

$$c_3 = 0$$

$$E_z = (c_1 \cos Bx + c_2 \sin Bx)c_4 \sin Ay \quad \text{—— (IV)}$$

(ii)Left conducting wall

$E_z = 0$ At x=0 $\forall y \rightarrow$ 0 to b

Apply above boundary condition in IV

TM wave

$$0 = c_1 c_4 \sin Ay$$

$$c_1 = 0$$

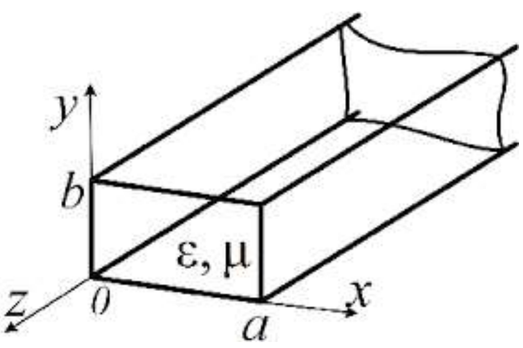
$$E_z = c_2 c_4 \sin Bx \sin Ay \qquad \text{—— (V)}$$

(iii)Top conducting wall

$E_z = 0$ At y=b $\forall x \rightarrow$ 0 to a

Apply above boundary condition in V

$$0 = c_2 c_4 \sin Bx \sin Ab$$



$$Sin Ab = 0$$

$$Ab = n\pi \qquad \text{n=0,1,2,3...}$$

$$A = \frac{n\pi}{b}$$

$$E_z = c \sin Bx \sin(\frac{n\pi}{b} y) \qquad \text{—— (VI)}$$

TM wave

(iv)Right conducting wall

$E_z = 0$ At x=a $\forall y \rightarrow$ 0 to b

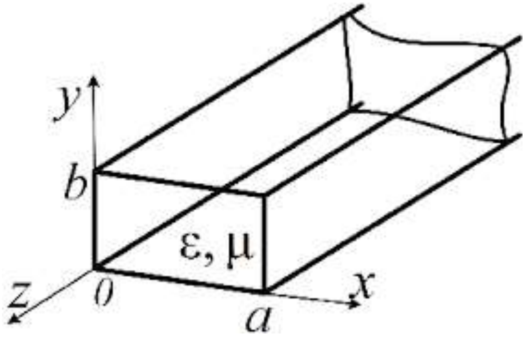
Apply above boundary condition in VI

$$0 = c\sin Ba \sin(\frac{n\pi}{b}y)$$

$$\sin Ba = 0$$

$$Ba = m\pi$$ m=0,1,2,3…

$$B = \frac{m\pi}{a}$$



$$E_z = c\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)e^{-\bar{\gamma}z}$$

TM wave

$$E_x = -\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial x} = \frac{-c\bar{\gamma}}{h^2}\frac{m\pi}{a}\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\,\mathrm{e}^{-\bar{\gamma}z}$$

$$E_y = -\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial y} = \frac{-c\bar{\gamma}}{h^2}\frac{n\pi}{b}\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\,\mathrm{e}^{-\bar{\gamma}z}$$

$$H_x = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y} = \frac{cj\omega\varepsilon}{h^2}\frac{n\pi}{b}\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\,\mathrm{e}^{-\bar{\gamma}z}$$

$$H_y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} = \frac{-cj\omega\varepsilon}{h^2}\frac{m\pi}{a}\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\,\mathrm{e}^{-\bar{\gamma}z}$$

For different values of integer m and n there are different field configurations and these field configurations are called as modes(TM_{mn} modes).
m represents field variation along X-direction, n represents field variation along Y-direction
For m=0,n=0; m=0,n=1; m=1,n=0 all the components are equal to zeros. i.e $\mathrm{TM}_{00,}\mathrm{TM}_{01,}\mathrm{TM}_{10}$ modes does not exist. Least possible mode in TM_{mn} waves is TM_{11}

TE wave

$$E_z = 0$$

$$\nabla^2 H = -\omega^2 \mu\varepsilon H$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu\varepsilon H_z$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = -(\overline{\gamma}^2 + \omega^2 \mu\varepsilon)\mathrm{H}_z$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = -h^2 H_z$$

Let $H_z = X(x)Y(y)$

TE wave

$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} = -h^2 XY$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = -h^2$$

Let $\frac{1}{X}\frac{d^2 X}{dx^2} = -B^2$ ——— (I) ; $\frac{1}{Y}\frac{d^2 Y}{dy^2} = -A^2$ ——— (II)

$$B^2 + A^2 = h^2$$

Solution of I and II are $X = c_1 \cos Bx + c_2 \sin Bx$

$$Y = c_3 \cos \mathrm{Ay} + c_4 \sin \mathrm{Ay}$$

TE wave

$$H_z = XY = (c_1 \cos Bx + c_2 \sin Bx)(c_3 \cos Ay + c_4 \sin Ay) \quad \text{——— (III)}$$

(i)Bottom conducting wall

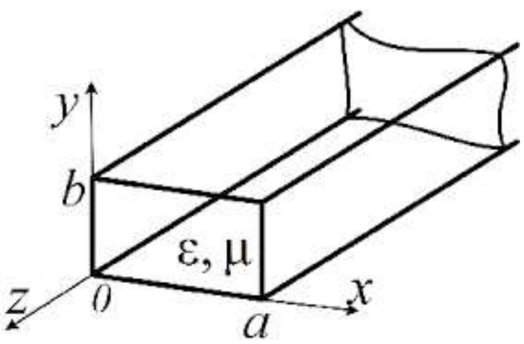
$\mathrm{E}_x = 0$ At y=0 $\forall x \rightarrow$ 0 to a

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} = \frac{-j\omega\mu}{h^2}(c_1 \cos Bx + c_2 \sin Bx)(-c_3 \mathrm{A}\sin\mathrm{A}y + c_4 \mathrm{A}\cos\mathrm{A}y) \quad \text{——— (IV)}$$

Apply above boundary condition in IV

$$0 = \frac{-j\omega\mu}{h^2}(c_1 \cos Bx + c_2 \sin Bx)c_4 \mathrm{A}$$

$$\mathrm{c}_4 = 0$$



$$H_z = (c_1 \cos Bx + c_2 \sin Bx)(c_3 \cos Ay)$$

TE wave

(ii)Left conducting wall

$$\mathrm{E}_y = 0 \quad \text{At x=0} \quad \forall y \rightarrow \text{0 to b}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} = \frac{j\omega\mu}{h^2}(-c_1\,\mathrm{B}\sin Bx + c_2\,\mathrm{B}\cos Bx)(c_3\cos \mathrm{Ay}) \qquad \text{(V)}$$

Apply above boundary condition in V

$$0 = \frac{j\omega\mu}{h^2}(c_2\,\mathrm{B}\cos Bx)(c_3\cos \mathrm{Ay})$$

$$c_2 = 0$$

$$H_z = c_1 c_3(\cos Bx)(\cos \mathrm{Ay})$$

(iii)Top conducting wall

$$\mathrm{E}_x = 0 \quad \text{At y=b} \quad \forall x \rightarrow \text{0 to a}$$

TE wave

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} = \frac{-j\omega\mu}{h^2} c_1 c_3 (\cos Bx)(-A\sin Ay) \qquad \text{——— (VI)}$$

Apply above boundary condition in VI

$$0 = \frac{-j\omega\mu}{h^2} c_1 c_3 (\cos Bx)(-A\sin Ab)$$

$$\sin Ab = 0$$

$$Ab = n\pi \qquad \text{n=0,1,2,3...}$$

$$A = \frac{n\pi}{b}$$

$$H_z = c(\cos Bx)(\cos \frac{n\pi}{b} y)$$

TE wave

(iv)Right conducting wall

$$E_y = 0 \quad \text{At x=a} \quad \forall y \to \text{0 to b}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} = \frac{j\omega\mu}{h^2}(-cB\sin Bx)(\cos\frac{n\pi}{b}\mathrm{y}) \qquad \text{——— (VII)}$$

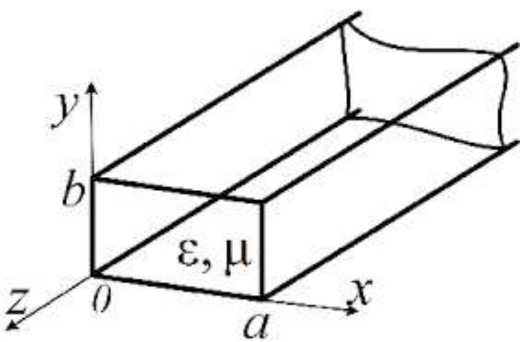
Apply above boundary condition in VII

$$0 = \frac{j\omega\mu}{h^2}(-cB\sin Ba)(\cos\frac{n\pi}{b}\mathrm{y})$$

$$\mathrm{sinBa} = 0$$

$$\mathrm{Ba} = \mathrm{m}\pi \qquad \text{m=0,1,2,3...}$$

$$B = \frac{m\pi}{a}$$



$$H_z = c\cos(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}\mathrm{y})\,\mathrm{e}^{-\bar{\gamma}z}$$

TE wave

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} = \frac{j\omega\mu}{h^2} c \frac{n\pi}{b} \cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}\mathrm{y})\,\mathrm{e}^{-j\bar{\beta}z}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} = \frac{-j\omega\mu}{h^2} c \frac{m\pi}{a} \sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}\mathrm{y})\,\mathrm{e}^{-j\bar{\beta}z}$$

$$H_x = \frac{-\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial x} = \frac{\bar{\gamma}}{h^2} c \frac{m\pi}{a} \sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}\mathrm{y})\,\mathrm{e}^{-j\bar{\beta}z}$$

$$H_y = \frac{\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial y} = \frac{\bar{\gamma}}{h^2} c \frac{n\pi}{b} \cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}\mathrm{y})\,\mathrm{e}^{-j\bar{\beta}z}$$

For different values of integer m and n there are different field configurations and these field configurations are called as modes(TE_{mn} modes).

m represents field variation along X-direction, n represents field variation along Y-direction

For m=0,n=0; all the components are equal to zeros. i.e TE_{00} mode does not exist.

For m=0, n=1; $E_x, H_y, H_z \neq 0$ For m=1, n=0; $E_y, H_x, H_z \neq 0$

Least possible mode in TE_{mn} waves are $TE_{10,}TE_{01}$

Cutoff frequency

$$\bar{\gamma}^2 + \omega^2 \mu\varepsilon = h^2$$

$$h^2 = B^2 + A^2$$

$$h^2 = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2$$

$$\bar{\gamma}^2 = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 - \omega^2 \mu\varepsilon$$

$$\bar{\gamma} = \sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 - \omega^2 \mu\varepsilon}$$

$$\omega^2 \mu\varepsilon > (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2$$

$$\bar{\gamma} = j\bar{\beta} = j\sqrt{\omega^2 \mu\varepsilon - [(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2]}$$

Cutoff frequency

$$\bar{\beta} = \sqrt{\omega^2 \mu\varepsilon - [(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2]}$$

$$At \omega = \omega_c \Rightarrow \bar{\beta} = 0$$

$$\omega_c^{\ 2} \mu\varepsilon = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2$$

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$$

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}$$

Characteristics of TE, TM waves

If the medium between two conducting plates is freespace

$$f_c = \frac{c}{2}\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}$$

If $\omega < \omega_c (f < f_c)$

$\bar{\gamma}$ Is purely real and there is no wave propagation

If $\omega > \omega_c (f > f_c)$

$\bar{\gamma}$ Is purely imaginary and there is wave propagation

To get wave propagation in the waveguide operating frequency f must be greater than cutoff frequency and we can say waveguide acting as High Pass Filter

Cutoff frequency

Cutoff frequency is defined as the frequency below which wave propagation ceases

$$f_c = \frac{c}{2}\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}$$

$$\lambda_c = \frac{c}{f_c}$$

Cutoff wave length $\lambda_c = \dfrac{2}{\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}}$

The mode which is having least cutoff frequency and highest cutoff wave length is called as dominant mode

$$f_{cTE_{10}} = \frac{c}{2a}$$

$$\lambda_{cTE_{10}} = 2a$$

Cutoff frequency

$$f_{cTE_{01}} = \frac{c}{2b}$$

$$f_{cTE_{11}} = \frac{c}{2}\sqrt{(\frac{1}{a})^2 + (\frac{1}{b})^2}$$

$$f_{cTM_{11}} = \frac{c}{2}\sqrt{(\frac{1}{a})^2 + (\frac{1}{b})^2}$$

In TE_{mn} waves dominant mode is TE_{10}

In TM_{mn} waves dominant mode is TM_{11}

TE_{10} is dominant mode for the entire rectangular waveguide

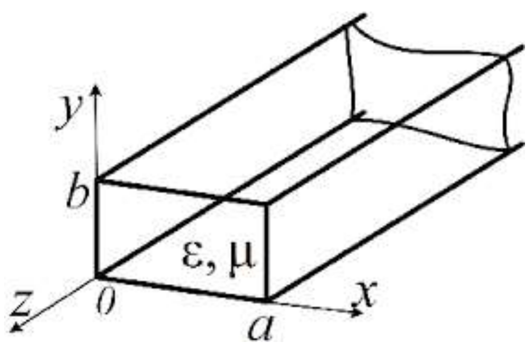
1. A rectangular waveguide has a=4cm, b=3cm as its sectional dimensions. Conclude all the modes which will propagate at 500MHz.

$$f_{cTE_{10}} = \frac{c}{2a} = \frac{3x10^{10}}{2x4} = 3.75\text{GHz}$$

To get wave propagation in the waveguide operating frequency f must be greater than cutoff frequency

$$f < f_c$$

No mode is possible



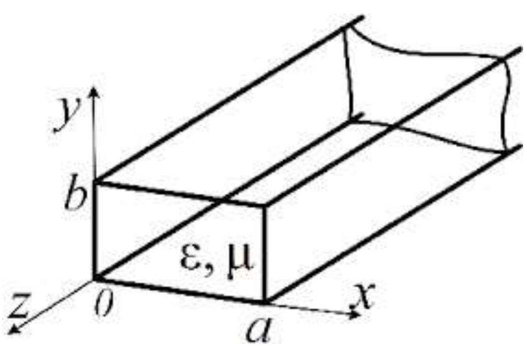
2. A Rectangular wave guide is filled by dielectric material of ε_r=9 and has dimensions of 7*3.5cm, it operates in the dominant TE mode. Determine cutoff frequency, phase velocity in guide at 2GHz.

Given a=7cm, b=3.5cm, f=2GHz
Dominant mode TE10

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}$$

$$f_{c_{TE10}} = \frac{1}{2a\sqrt{\mu_0\varepsilon_0\varepsilon_r}} = \frac{c}{2a\sqrt{\varepsilon_r}} = \frac{3x10^{10}}{2x7x\sqrt{9}}$$ =0.71GHz

$$\overline{v} = \frac{c}{\sqrt{(1-\frac{f_c^{\,2}}{f^{\,2}})}} = 3.73x10^8 m/s$$



3. A standard air-filled waveguide with dimensions a=8.636cm, b=4.318cm is fed by a 4GHz carrier from a coaxial cable. Justify whether a TE_{10} mode will be propagated. If so, calculate the phase velocity and group velocity

$$f_{cTE_{10}} = \frac{c}{2a} = \frac{3x10^{10}}{2x8.636}$$ =1.73GHz

f=4GHz

$$f > f_c$$

SoTE10 mode is possible

$$\overline{v} = \frac{c}{\sqrt{(1 - \frac{f_c^{\,2}}{f^{\,2}})}} = 3.38x10^8 m/s$$

$$v_g = c\sqrt{(1 - \frac{f_c^{\,2}}{f^{\,2}})} \quad = 2.84x10^8 m/s$$

4. An air field rectangular wave guide has dimensions of a = 6 cm, b=4cm.The signal frequency is 3 GHz. Calculate Cut off frequency, Wave length, phase velocity for TE_{10}, TE_{11} modes

$$f_{cTE_{10}} = \frac{c}{2a} = \frac{3x10^{10}}{2x6}$$ =2.5GHZ

$$\lambda_{cTE_{10}} = 2a$$ =12cm

$$f_{cTE11} = \frac{c}{2}\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2} = \frac{c}{2}\sqrt{(\frac{1}{a})^2 + (\frac{1}{b})^2}$$ =4.5GHz

$$f < f_{cTE11}$$

$$= 5.42x10^{10}$$

$$f < f_{cTE11}$$ So TE11 mode is not possible

$$\bar{v} = \frac{c}{\sqrt{(1 - \frac{f_c^{\ 2}}{f^{\ 2}})}} = 5.42x10^{10}$$ m/s

5.A 6GHz signal is to be propagated in the dominant mode in a RWG in its group velocity Is to be 80% of c. What must be the width of WG. What impedance will it offer to this if it is correctly matched.

$$v_g = c\sqrt{(1-\frac{f_c^2}{f^2})}$$

$$f_{cTE_{10}} = \frac{c}{2a}$$

$$Z_{TE} = \frac{\eta}{\sqrt{1-\frac{f_c^2}{f^2}}}$$

Unit-IV
Circular wave guide

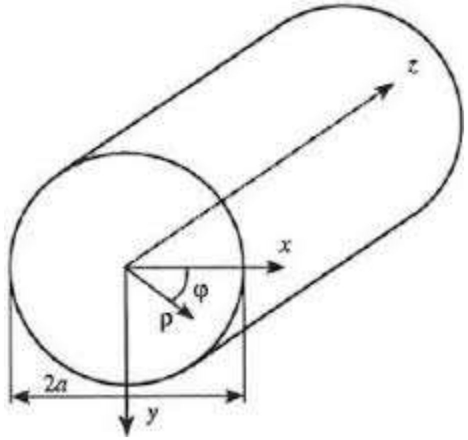
Contents

1.General field equations

2.Transverse Magnetic waves(TM Waves)

3. Transverse Electric waves(TE Waves)

4.Cutoff frequency



Unit-IV
Rectangular cavity Resonator

Contents

1.TE,TM Waves

2.Resonant frequency

3.Q factor

Unit-V
Cylindrical cavity Resonator

Contents

1.TE,TM Waves

2.Resonant frequency

3.Q factor

Unit-V
Microstrip lines

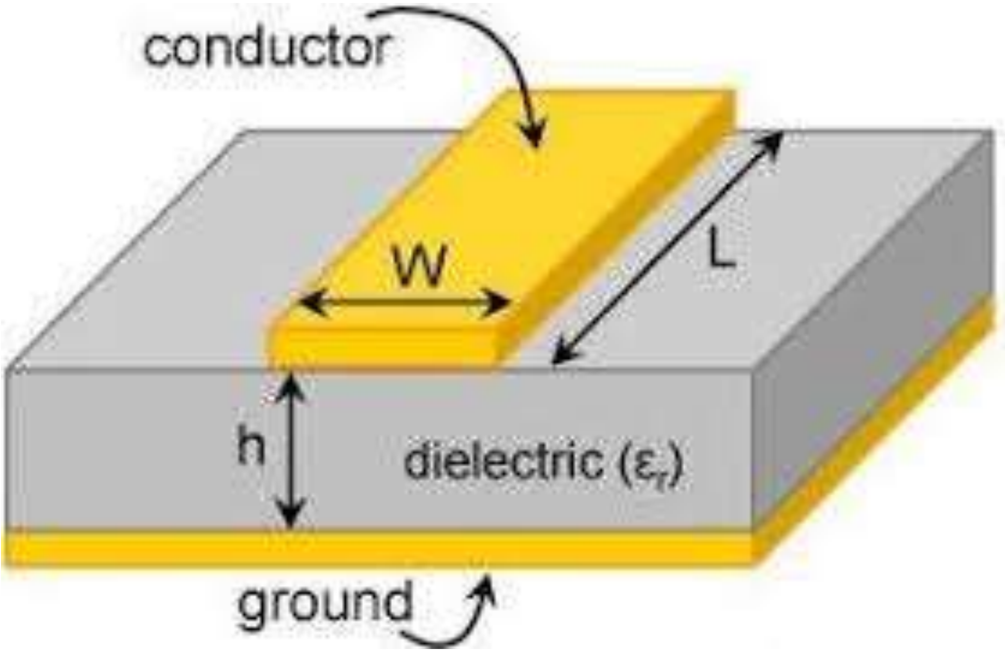
Contents

1.Structure

2.Characteristic impedance

3.Losses

4.Q factor



Circular Waveguide

In solving for the electromagnetic fields with in guides of circular cross section, a differential equation known as Bessel's equation is encountered.

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(1 - \frac{n^2}{\rho^2}\right)P = 0 \rightarrow \text{Bessel's equation}$$

The solution of the equation leads to Bessel's function.

$J_n(\rho) \rightarrow$ Bessel's function of first kind order n.

These same functions can be expected to appear in any two dimensional problem in which there is circular symmetry. Example of such problem is the propagation of waves within a circular cylindrical waveguide.

General equations:

The method of solution of the electromagnetic equations for guides of circular cross section is similar to that followed for rectangular guides.

$$\nabla \times E = -j\omega\mu H \quad -(1)$$
$$\nabla \times H = j\omega\epsilon E \quad -(2)$$

} Maxwell's equations for nonconducting region.

$$(1) \Rightarrow \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ E_\rho & \rho E_\phi & E_z \end{vmatrix} = -j\omega\mu(H_\rho a_\rho + H_\phi a_\phi + H_z a_z)$$

~~Compare~~ a_ρ, a_ϕ, a_z coefficients

$$\frac{\partial E_z}{\partial\phi} - \frac{\partial(\rho E_\phi)}{\partial z}$$

Let us assume that along ρ, ϕ directions there is variation of field components along z - there is wave propagation along the guide

i.e $\frac{\partial}{\partial z} \rightarrow -\bar{\gamma}$

$$\frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & -\bar{\gamma} \\ E_\rho & \rho E_\phi & E_z \end{vmatrix} = -j\omega\mu(H_\rho a_\rho + H_\phi a_\phi + H_z a_z)$$

Now compare a_ρ, a_ϕ, a_z coefficients

$$\frac{1}{\rho}\frac{\partial}{\partial \phi}E_z + \bar{\gamma}E_\phi = -j\omega\mu H_\rho \quad -(3)$$

$$-\bar{\gamma}E_\rho - \frac{\partial E_z}{\partial \rho} = -j\omega\mu H_\phi \quad -(4)$$

$$\frac{1}{\rho}\left[\frac{\partial(\rho E_\phi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi}\right] = -j\omega\mu H_z \quad -(5)$$

$$(2) \Rightarrow \frac{1}{\rho}\begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & -\bar{\gamma} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix} = j\omega\epsilon(E_\rho a_\rho + E_\phi a_\phi + E_z a_z)$$

$$\frac{\partial H_z}{\rho \partial \phi} + \bar{\gamma}H_\phi = j\omega\epsilon E_\rho \quad -(6)$$

$$-\bar{\gamma}H_\rho - \frac{\partial H_z}{\partial \rho} = j\omega\epsilon E_\phi \quad -(7)$$

$$\frac{1}{\rho}\left[\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi}\right] = j\omega\epsilon E_z \quad -(8)$$

These (3) to (6) equations can be combined to give

$$\Rightarrow \quad E_\rho = \frac{-\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial \rho} - \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\rho\partial \phi}$$

$$E_\phi = \frac{-\bar{\gamma}}{h^2}\frac{\partial E_z}{\rho\partial \phi} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial \rho}$$

$$H_\rho = \frac{-\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial \rho} + \frac{j\omega\epsilon}{h^2}\frac{\partial E_z}{\rho\partial \phi}$$

$$H_\phi = \frac{-\bar{\gamma}}{h^2}\frac{\partial H_z}{\rho\partial \phi} - \frac{j\omega\epsilon}{h^2}\frac{\partial E_z}{\partial \rho}$$

T.E. waves & TM waves

The wave equation in cylindrical coordinates for E_z is

$$\nabla^2 E_z = -\omega^2 \mu \epsilon E_z$$

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu \epsilon E_z$$

Let $E_z = P(\rho) Q(\phi) e^{-\bar{\gamma} z}$

where $P(\rho)$ is a function of ρ alone and $Q(\phi)$ is a function of ϕ alone. Substituting the expression for E_z in the wave equation gives

$$Q \frac{d^2 P}{d\rho^2} + \frac{Q}{\rho}\frac{dP}{d\rho} + \frac{P}{\rho^2}\frac{d^2 Q}{d\phi^2} + PQ\bar{\gamma}^2 = -\omega^2 \mu \epsilon PQ \quad \left[\because \frac{\partial}{\partial z} = -\bar{\gamma}\right]$$

Divide by PQ

$$\frac{1}{P}\frac{d^2 P}{d\rho^2} + \frac{1}{\rho P}\frac{dP}{d\rho} + \frac{1}{Q\rho^2}\frac{d^2 Q}{d\phi^2} = -h^2 \quad \left[\because h^2 = \bar{\gamma}^2 + \omega^2 \mu \epsilon\right]$$

Let $\frac{1}{Q}\frac{d^2 Q}{d\phi^2} = -n^2 \Rightarrow Q = (A_n \cos n\phi + B_n \sin n\phi)$

$$\frac{d^2 P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(h^2 - \frac{n^2}{\rho^2}\right)P = 0$$

where n is a constant

Dividing through by h^2 above equation transformed into

$$\frac{d^2 P}{d(\rho h)^2} + \frac{1}{\rho h}\frac{dP}{d(\rho h)} + \left[1 - \frac{n^2}{(\rho h)^2}\right]P = 0$$

It is standard form of Bessel's equation interms of (ρh).

$$P(\rho h) = J_n(\rho h)$$

$$E_z = J_n(\rho h)(A_n \cos n\phi + B_n \sin n\phi) e^{-\bar{\gamma} z}$$

The solution for H_z will have exactly the same form as for E_z and can therefore be written

$$H_z = J_n(\rho h)(C_n \cos n\phi + D_n \sin n\phi) e^{-\bar{\gamma} z}$$

Now $E_z = J_n(\rho h)(A_n \cos n\varphi + B_n \sin n\varphi)$

$$E_z = J_n(\rho h)\sqrt{A_n^2+B_n^2}\left[\frac{A_n}{\sqrt{A_n^2+B_n^2}}\cos n\varphi + \frac{B_n}{\sqrt{A_n^2+B_n^2}}\sin n\varphi\right]$$

$$E_z = J_n(\rho h)\, c\left[\cos\left(n\varphi + \tan^{-1}\left(\frac{B_n}{A_n}\right)\right)\right]$$

$$E_z = c J_n(\rho h)\cos n\varphi'$$

similarly $H_z = c J_n(\rho h)\cos n\varphi'$

For TM waves ($H_z = 0$, $E_z \neq 0$)

$$E_\rho = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial \rho} = -\frac{i c \omega \epsilon n}{h^2 \rho} J_n(\rho h)\sin n\varphi'$$

$$E_\rho = -\frac{i\beta}{h^2} c J_n'(\rho h)\, h \cos n\varphi' = -\frac{i\beta c}{h} J_n'(\rho h)\cos n\varphi'$$

$$E_\varphi = -\frac{\gamma}{h^2}\frac{\partial E_z}{\rho\,\partial\varphi} = +\frac{i\beta c}{h^2}\frac{J_n(\rho h)}{\rho}\, n \sin n\varphi'$$

$$H_\rho = \frac{i c \omega \epsilon}{h^2}\frac{\partial E_z}{\partial \rho} = -\frac{i \omega \epsilon n}{h^2 \rho} J_n(\rho h)\sin n\varphi'$$

$$H_\varphi = -\frac{i c \omega \epsilon}{h} J_n'(\rho h)\cos n\varphi'$$

For TE waves ($E_z = 0$, $H_z \neq 0$)

$$E_\rho = -\frac{i\omega\mu}{h^2}\frac{\partial H_z}{\rho\,\partial\varphi} = -\frac{i\omega\mu}{h^2}\, c J_n'(\rho h)\, h \cos n\varphi'$$

$$E_\rho = -\frac{i\omega\mu}{h} c J_n'(\rho h)\cos n\varphi' \quad + \frac{i\omega\mu}{h^2}\frac{1}{\rho} J_n(\rho h)\, n \sin n\varphi'$$

$$E_\varphi = \frac{i\omega\mu}{h^2}\frac{\partial H_z}{\partial\rho} = \frac{i\omega\mu}{h^2} c J_n'(\rho h)\, h \cos n\varphi'$$

$$H_\rho = -\frac{i\beta c}{h} J_n'(h\rho)\cos n\varphi'$$

$$H_\varphi = \frac{i n \beta c}{h^2 \rho} J_n(h\rho)\sin n\varphi'$$

For TM waves ($H_z = 0$)

$E_z = J_n(\rho h)$ [redacted] $e^{\ldots}$ comp'

The boundary conditions require that E_z must vanish at the surface of the guide

$\therefore \quad J_n(ha) = 0$

Where a is the radius of the guide. There is an infinite number of possible TM waves corresponding to the infinite number of roots of $J_n(ha) = 0$.

The first few roots are

$(ha)_{01} = 2.405$ $\qquad$ $(ha)_{11} = 3.85$

$(ha)_{02} = 5.52$ $\qquad$ $(ha)_{12} = 7.02$

The first subscript (n) refers to no. of cycles along ϕ direction. The second (m) refers to the no. of zeros of the field along ρ-direction.

For TE waves ($E_z = 0$)

$H_z = J_n(\rho h) e$ comp'

The boundary conditions to met for TE waves are that $E_\phi = 0$ at $\rho = a$

i.e $E_\phi \propto \frac{\partial H_z}{\partial \rho}$ $\quad \therefore \; J_n'(ah) = 0$

$\therefore$ For TE waves the boundary conditions require that $J_n'(ah) = 0$

The first few roots are

$(ha)'_{01} = 3.83$ $\qquad$ $(ha)'_{11} = 1.84$

$(ha)'_{02} = 7.02$ $\qquad$ $(ha)'_{12} = 5.33$

cutoff frequency

$$h^2 = \bar{\gamma}^2 + \omega^2 \mu \epsilon$$

$$\bar{\gamma} = \sqrt{h^2 - \omega^2 \mu \epsilon}$$

$$\bar{\beta} = \sqrt{\omega^2 \mu \epsilon - h^2}$$

The cutoff or critical frequency below which transmission of a wave will not occur is

$$f_c = \frac{h}{2\pi\sqrt{\mu\epsilon}}$$

For TM waves $ah = (ah)_{nm}$

$$\Downarrow$$

$$h = \frac{(ah)_{nm}}{a}$$

$$f_c = \frac{(ah)_{nm}}{2\pi a\sqrt{\mu\epsilon}} \Rightarrow \lambda_c = \frac{2\pi a}{(ah)_{nm}}$$

For TE waves $(ah)' = (ah)'_{nm}$

$$\Downarrow$$

$$h = \frac{(ah)'_{nm}}{a}$$

$$f_c = \frac{(ah)'_{nm}}{2\pi a\sqrt{\mu\epsilon}} \Rightarrow \lambda_c = \frac{2\pi a}{(ah)'_{nm}}$$

Dominant mode

TE_{11} → Dominant mode [∵ from $f_c = \frac{(ah)'_{nm}}{2\pi a\sqrt{\mu\epsilon}}$]

TM_{01} → Next dominant mode

Degenerate modes

(TM_{02}, TE_{02}), (TM_{11}, TE_{01})

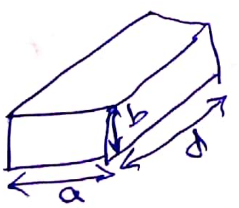
Cavity Resonators

(1)

When one end of the waveguide is terminated in a shorting plate there will be reflections and hence standing waves occur. When another shorting plate is kept then the hollow space so formed can support a signal which bounces back and forth between the two shorting plates. This results in resonance and hence the hollow space is called cavity and the resonator is called as the cavity resonator.

A cavity resonator is a metallic enclosure that confines the electromagnetic energy. The stored electric and magnetic energies inside the cavity determine its equivalent inductance and capacitance. The energy dissipated by the finite conductivity of the cavity walls determines its equivalent resistance. In practice the rectangular cavity resonator, circular cavity resonator, and reentrant cavity resonator are commonly used in many microwave applications.

Theoretically a given resonator has an infinite number of resonant modes, and each mode corresponds to a definite resonant frequency. When the frequency of an impressed signal is equal to a resonant frequency a maximum amplitude of the standing wave occurs and the peak energies stored in the electric and magnetic fields are equal. The mode having the lowest resonant frequency is known as the dominant mode.



$(d > a > b)$

Fig: Rectangular cavity Resonator

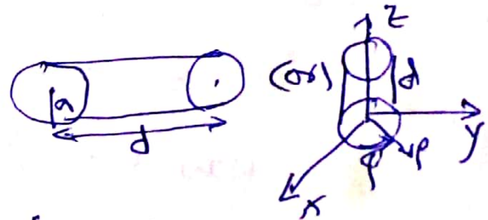
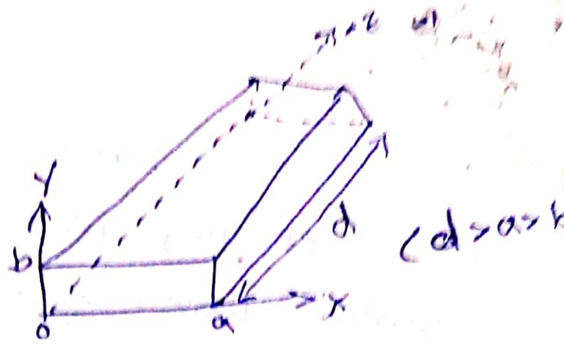


Fig: Circular cavity Resonator

Rectangular cavity resonator



$(d > a > b)$

The electromagnetic field inside the cavity should satisfy Maxwell's equations, subject to the boundary conditions that the electric field tangential to and the magnetic field normal to the metal walls must vanish.

It is necessary to choose the harmonic functions in z to satisfy above conditions at the remaining two end walls.

TMmnp waves ($H_z = 0, \; E_z \neq 0$)

$$\nabla^2 E_z = -\omega^2 \mu \epsilon E_z$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu \epsilon E_z$$

Let $E_z = X(x) Y(y) Z(z), \quad \omega^2 \mu \epsilon = h^2$

$$\therefore \; YZ \frac{\partial^2 X}{\partial x^2} + XZ \frac{\partial^2 Y}{\partial y^2} + XY \frac{\partial^2 Z}{\partial z^2} = -h^2 XYZ$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -h^2$$

Let $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -B^2$ $\qquad \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -C^2$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial z^2} = -A^2$$

$$B^2 + A^2 + C^2 = h^2$$

$$\therefore \; E_z = (C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos Ay + C_4 \sin Ay)(C_5 \cos Cz + C_6 \sin Cz)$$

After appling four boundary ~~condition~~ conditions to bottom, left, top, right conducting walls

$$E_z = E_{oz} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)(C_5 \cos Cz + C_6 \sin Cz)$$

Now for front end wall $E_x = 0$ at $z = 0 \;\; \forall \; x \to 0 \text{ to } a, \; y \to 0 \text{ to } b$

$$E_x = \frac{-j}{h^2}\frac{\partial E_z}{\partial x}$$

$$E_x = \frac{-\gamma}{h^2} E_{oz} \frac{m\pi}{a} \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)(C_5 \cos cz + C_6 \sin cz)$$

But $\gamma = \frac{\partial}{\partial z}$

$$E_x = \frac{+E_{oz}}{h^2}\left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)(-C_5 c \sin cz + C_6 c \cos cz)$$

$\therefore 0 = C_6$ $[\because E_x = 0$ at $z = 0]$

Now $E_z = E_{oz} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) C_5 \cos cz$

For back end wall $E_x = 0$ at $z = d$ $\forall$ $x \to 0$ to a, $y \to 0$ to b

$$E_x = \frac{-\gamma}{h^2} \frac{\partial E_z}{\partial x}$$

$$E_x = E_{oz} \frac{m\pi}{a} \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) C_5 \cos cz$$

But $\gamma = \frac{\partial}{\partial z}$

$$E_x = +E_{oz} \frac{m\pi}{a} \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) C_5 c \sin cz$$

$0 = \sin cd$ $[\because E_x = 0$ at $z = d]$

$\Downarrow$

$cd = p\pi$; $p = 0, 1, 2 \ldots$

$\Downarrow$

$$\boxed{c = \frac{p\pi}{d}}$$

$$\boxed{E_z = E_{oz} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right)}$$

where $m = 1, 2, 3 \ldots$ represents the number of half wave cycles in the x-direction

$n = 1, 2, 3 \ldots$ represents the number of half wave cycles in the y-direction

$p = 0, 1, 2, 3 \ldots$ represents the number of half wave cycles in the z-direction

TEmnp waves ($E_z = 0$, $H_z \neq 0$)

$$\nabla^2 H_z = -\omega^2 \mu\epsilon H_z$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu\epsilon H_z$$

Let $H_z = X(x) Y(y) Z(z)$

$\therefore H_z = (C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos Ay + C_4 \sin Ay)(C_5 \cos Cz + C_6 \sin Cz)$

After applying four boundary conditions to bottom, left, top, right conducting walls

$$H_z = H_{oz} \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)(C_5 \cos Cz + C_6 \sin Cz)$$

Now for front end wall $H_z = 0$ at $z = 0$ $\forall$ $x \to 0$ to a, $y \to 0$ to b

$$0 = C_5$$

Now $H_z = H_{oz} \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)(C_6 \sin Cz)$

For backend wall $H_z = 0$ at $z = d$ $\forall$ $x \to 0$ to a, $y \to 0$ to b

$$0 = \sin Cd \Rightarrow Cd = p\pi; \quad p = 1, 2, 3 \ldots$$

$$\Downarrow$$

$$\boxed{C = \frac{p\pi}{d}}$$

$$\boxed{H_z = H_{oz} \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right)}$$

Resonant frequency:

The frequency at which peak values of fields occur and peak energies stored in electric and magnetic fields are equal is called as resonant frequency

$$h^2 = B^2 + A^2 + C^2$$

$$\omega_r^2 \mu\epsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2$$

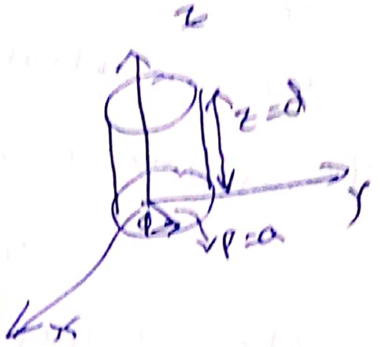
$$\omega_r = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

$$f_r = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2+\left(\frac{p}{d}\right)^2}$$

Dominant mode: The mode which has lowest resonant frequency is called as dominant mode.

TE_{101} → Dominant mode.

Circular cavity resonator



TM_{nmp} waves ($H_z = 0$, $E_z \neq 0$)

$$\nabla^2 E_z = -\omega^2 \mu\epsilon E_z$$

$$E_z = C J_n(\rho h)\cos n\phi\,(C_5 \cos cz + C_6 \sin cz)$$

$[\because E_z = P(\rho)Q(\phi)Z(z)]$

Now for bottom wall $E_\rho = 0$ at $z = 0$ ∀ $\rho \to 0$ to a, $\phi \to 0$ to 2π

$$E_\rho = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial \rho} = \frac{-\gamma}{h^2}\cdot C J_n'(\rho h)\,h\cos n\phi\,(C_5 \cos cz + C_6 \sin cz)$$

But $-\gamma = \frac{\partial}{\partial z}$

$$E_\rho = \frac{c}{h^2} J_n'(\rho h)\cdot h\cos n\phi\,(-C_5 c\sin cz + C_6 c\cos cz)$$

$0 = C_6$ $[\because E_\rho = 0$ at $z = 0)$

$\therefore$ $E_z = C J_n(\rho h)\cos n\phi\, C_5 \cos cz$

For top wall $E_\rho = 0$ at $z = d$

$$E_\rho = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial \rho} = \frac{-\gamma}{h^2} C J_n'(\rho h)\, h\cos n\phi\, C_5 \cos cz$$

But $-\gamma = \frac{\partial}{\partial z}$

$$E_\rho = -\frac{C J_n'(\rho h)h}{h^2}\cos n\phi\, C_5\, c\sin cz$$

$0 = \sin cd$ $[\because E_\rho = 0$ at $z = d]$

⇓

$cd = p\pi$; $p = 0, 1, 2 \ldots$

⇓ $c = \frac{p\pi}{d}$

$$E_z = E_{oz} J_n(\rho h) \cos n\phi \cos\left(\frac{p\pi}{d} z\right)$$ $\left[\because h = \frac{(ah)_{nm}}{a}\right]$

Where $n = 0, 1, 2, 3 \ldots$ is the number of the cycles in the ϕ-direction

$m = 1, 2, 3, \ldots$ is the number of zeros of the field in the radial direction

$p = 0, 1, 2, 3 \ldots$ is the number of halfwaves in the axial direction

J_n = Bessel's function of the first kind

E_{oz} = Amplitude of the electric field.

TE_{nmp} waves ($E_z = 0$, $H_z \neq 0$)

$$\nabla^2 H_z = -\omega^2 \mu \epsilon H_z$$

$$H_z = c J_n(\rho h) \cos n\phi (C_5 \cos cz + C_6 \sin cz)$$

For bottom wall at $z = 0$; $H_z = 0$

$$0 = C_5$$

$$H_z = c J_n(\rho h) \cos n\phi \, C_6 \sin cz$$

For top wall $H_z = 0$ at $z = d$

$$0 = \sin cd \Rightarrow cd = p\pi \Rightarrow c = \frac{p\pi}{d}$$

$p = 1, 2, 3 \ldots$

$$H_z = H_{oz} J_n(\rho h) \cos n\phi \sin\left(\frac{p\pi}{d} z\right)$$

Resonant ~~Cutoff~~ frequency

TM_{nmp} waves

$$h^2 = \left(\left(\frac{(ah)_{nm}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right)$$

$$\omega_r^2 \mu \epsilon = \left(\left(\frac{(ah)_{nm}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right)$$

$$\omega_r = \frac{1}{\sqrt{\mu \epsilon}} \sqrt{\left(\frac{(ah)_{nm}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

$$f_r = \frac{1}{2\pi \sqrt{\mu \epsilon}} \sqrt{\left(\frac{(ah)_{nm}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

TE_{mnp} waves

$$h^2 = \left(\frac{(\alpha h)'_{nm}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2$$

$$\omega_r^2 \mu\epsilon = \left(\frac{(\alpha h)'_{nm}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2$$

$$\omega_r = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{(\alpha h)'_{nm}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

$$\boxed{f_r = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{(\alpha h)'_{nm}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}}$$

Dominant mode

TM_{110} — for $d < 2a$

TE_{111} — for $d \geq 2a$

Q factor of Rectangular cavity resonator (UNIT-IV)

Quality factor(Q) is the measure of frequency selectivity of resonator or measure of loss in the resonator

$$Q = \omega_r \frac{w}{p_{loss}}$$

Where ω_r is resonant frequency

w is energy stored in the resonator

p_{loss} is the power loss in the resonator

Energy density stored in the form of electric field $W_e = \frac{1}{2}\varepsilon E^2$

Energy density stored in the form of magnetic field $W_m = \frac{1}{2}\mu H^2$

At resonance $w_e = w_m = w$

Q factor of Rectangular cavity resonator (UNIT-IV)

$$TE_{101}(E_y, H_x, H_z \neq 0)$$

Energy stored in the form of electric field $w_e = \frac{1}{2}\int \varepsilon E^2 dv$

$$w_e = \frac{1}{2}\int_0^a\int_0^b\int_0^d \varepsilon E_y^{\,2} dxdydz$$

Energy stored in the form of magnetic field $w_m = \frac{1}{2}\int \mu H^2 dv$

$$w_m = \frac{1}{2}\int_0^a\int_0^b\int_0^d \mu(\mathrm{H}_x^{\,2} + H_z^{\,2})dxdydz$$

Power loss per unit area due to conducting wall $P_{loss} = \frac{1}{2}R_s \, |\, \mathrm{J}_s \,|^2$

$$H_t = J_s$$

$$P_{loss} = \frac{1}{2}R_s \, |\, H_t \,|^2$$

Q factor of Rectangular cavity resonator (UNIT-IV)

Power loss due to conducting wall $$p_{loss}=\frac{1}{2}\int R_s\,|H_t|^2\,ds$$

Power loss due to front and back conducting walls

$$p_{loss}=\frac{R_s}{2}\left(\int_0^a\int_0^b |H_x(z=0)|^2\,dxdy+\int_0^a\int_0^b |H_x(z=d)|^2\,dxdy\right) \quad (1)$$

Power loss due to left and right conducting walls

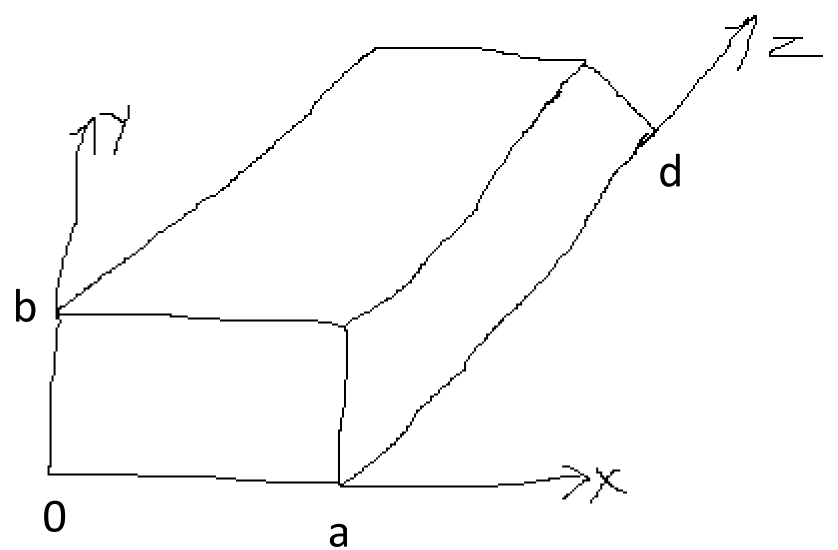
$$p_{loss}=\frac{R_s}{2}\left(\int_0^b\int_0^d |H_z(x=0)|^2\,dydz+\int_0^b\int_0^d |H_z(x=a)|^2\,dydz\right) \quad (2)$$

Power loss due to top and bottom conducting walls

$$p_{loss}=\frac{R_s}{2}\left(\int_0^a\int_0^d (|H_x(y=0)|^2+|H_z(y=0)|^2)dxdz+\int_0^a\int_0^d (|H_x(y=b)|^2+|H_z(y=b)|^2)dxdz\right) \quad (3)$$

Q factor of Rectangular cavity resonator (UNIT-IV)

By adding equations (1), (2) and (3) total power loss in the resonator is obtained



Q factor of cylindrical cavity resonator (UNIT-V)

Quality factor(Q) is the measure of frequency selectivity of resonator or measure of loss in the resonator

$$Q = \omega_r \frac{w}{p_{loss}}$$

Where ω_r is resonant frequency

w is energy stored in the resonator

p_{loss} is the power loss in the resonator

Energy density stored in the form of electric field $W_e = \frac{1}{2}\varepsilon E^2$

Energy density stored in the form of magnetic field $W_m = \frac{1}{2}\mu H^2$

At resonance $w_e = w_m = w$

Q factor of cylindrical cavity resonator (UNIT-V)

$$TE_{111}(E_\rho, E_\phi, H_\rho, H_\phi, H_z \neq 0)$$

Energy stored in the form of electric field $w_e = \frac{1}{2}\int \varepsilon E^2 dv$

$$w_e = \frac{\varepsilon}{2}\int_0^a \int_0^{2\pi} \int_0^d ({E_\rho}^2 + {E_\phi}^2)\rho d\rho d\phi d\,z$$

Energy stored in the form of magnetic field $w_m = \frac{1}{2}\int \mu H^2 dv$

$$w_m = \frac{\mu}{2}\int_0^a \int_0^{2\pi} \int_0^d ({\mathrm{H}_\rho}^2 + {\mathrm{H}_\phi}^2 + {H_z}^2)\rho d\rho d\phi d\,z$$

Power loss per unit area due to conducting wall $P_{loss} = \frac{1}{2} R_s \, |\, \mathrm{J}_s \,|^2$

$$H_t = J_s$$

$$P_{loss} = \frac{1}{2} R_s \, |\, H_t \,|^2$$

Q factor of cylindrical cavity resonator (UNIT-V)

Power loss due to conducting wall $$p_{loss} = \frac{1}{2}\int R_s \, |H_t|^2 \, ds$$

Power loss due to bottom and top conducting walls

$$p_{loss} = \frac{R_s}{2}\left(\int_0^a \int_0^{2\pi} (|H_\rho(z=0)|^2 + |H_\phi(z=0)|^2)\rho d\rho d\phi + \int_0^a \int_0^{2\pi} (|H_\rho(z=d)|^2 + |H_\phi(z=d)|^2)\rho d\rho d\phi\right)$$

(1)

Power loss due to cylindrical conducting wall

$$p_{loss} = \frac{R_s}{2}\left(\int_0^{2\pi} \int_0^{d} |H_z(\rho=a)|^2 \, \rho d\phi dz + \int_0^{2\pi} \int_0^{d} |H_\phi(\rho=a)|^2 \, \rho d\phi dz\right) \quad (2)$$

By adding equations (1), (2) total power loss in the resonator is obtained

Q factor of cylindrical cavity resonator (UNIT-V)

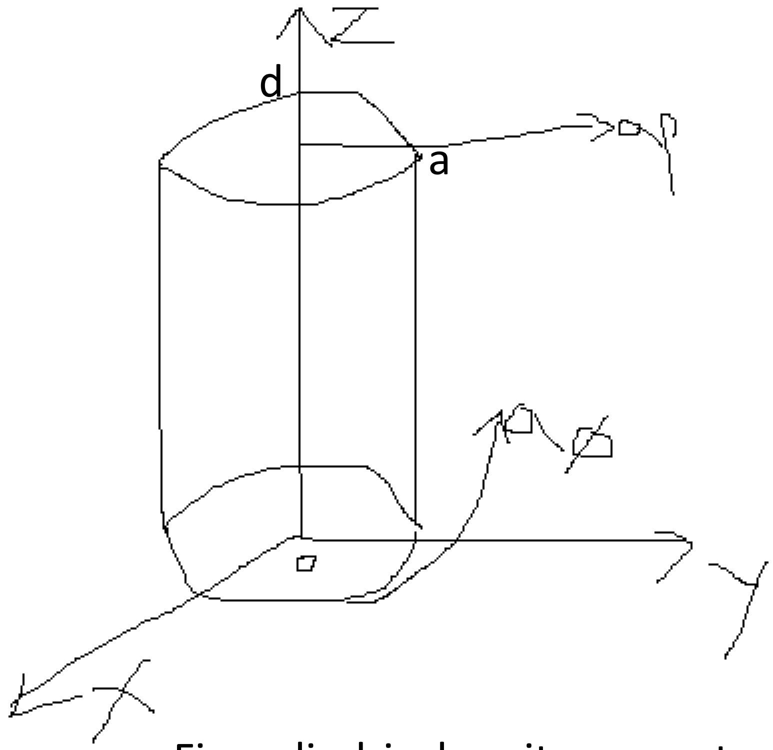


Fig: cylindrical cavity resonator

1.A rectangular cavity resonator has the following dimension: a=5cm, b=2cm and d=15cm. Calculate (i) the resonant frequency of the dominant mode TE_{101} for an air filled cavity (ii) dielectric filled cavity of εr=4.

$$f_r = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{((\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 + (\frac{p\pi}{d})^2)}$$

(i) $$f_{r_{TE101}} = \frac{c}{2}\sqrt{(\frac{1}{a})^2 + (\frac{1}{d})^2}$$ =3.14GHz

(ii) $$f_{r_{TE101}} = \frac{c}{2\sqrt{\varepsilon_r}}\sqrt{(\frac{1}{a})^2 + (\frac{1}{d})^2}$$ =1.57GHz

2.Calculate the resonant frequency of a circular resonator of following dimensions. Diameter=12.5cm and length=5cm for TM_{012} mode.

$$f_r = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{(\frac{(ah)_{nm}}{a})^2 + (\frac{p\pi}{d})^2}$$

$$f_r = \frac{c}{2\pi}\sqrt{(\frac{(ah)_{nm}}{a})^2 + (\frac{p\pi}{d})^2}$$

2a=12.5cm

d=5cm

n=0, m=1, p=2

$$(ah)_{01} = 2.405$$

fr=6.15GHz

1.An air filled circular waveguide has a radius of 2cm and is to carry energy at a frequency of 8GHz. Find all the TE and TM modes for which energy transmission is possible.

$$f_{c_{TE_{nm}}} = \frac{(ah)'_{nm}}{2\pi a\sqrt{\mu\varepsilon}}$$

$$f_{c_{TE_{11}}} = \frac{(ah)'_{11}}{2\pi a\sqrt{\mu\varepsilon}}$$

$$f_{c_{TE_{11}}} = \frac{1.84c}{2\pi a}$$ =4.3GHz

Operating frequency f = 8GHz

$$f > f_{c_{TE_{11}}} \Rightarrow TE_{11}$$ Mode is possible

$$f_{c_{TM_{nm}}} = \frac{(ah)_{nm}}{2\pi a\sqrt{\mu\varepsilon}}$$

$$f_{c_{TM_{01}}} = \frac{(ah)_{01}}{2\pi a\sqrt{\mu\varepsilon}}$$

$$f_{c_{TM_{01}}} = \frac{2.405c}{2\pi a}$$ =5.7GHz

$$f > f_{c_{TM\,01}} \Rightarrow TM_{01}$$ Mode is possible

$$f_{c_{TE_{01}}} = \frac{3.83c}{2\pi a}$$ =9.14GHz

$$f < f_{c_{TE_{01}}} \Rightarrow TE_{01}$$ Mode is not possible

Possible modes are TE11 and TM01

2.A circular wave guide has a cut off frequency of 9GHz in dominant mode. Find the inside diameter of the guide if it is i) air-filled. ii) Filled with dielectric with ε_r=4.

$$f_{c_{TE_{11}}} = \frac{1.84c}{2\pi a}$$

2a=1.95cm

$$f_{c_{TE_{11}}} = \frac{(ah)'_{11}}{2\pi a\sqrt{\mu\varepsilon}}$$

$$f_{c_{TE_{11}}} = \frac{1.84}{2\pi a\sqrt{\mu_0\varepsilon_0\varepsilon_r}}$$

$$f_{c_{TE_{11}}} = \frac{1.84c}{2\pi a\sqrt{\varepsilon_r}}$$

2a= 0.98cm

3.An air filled circular waveguide is to be operated at a frequency of 6GHz and is to have dimensions such that fc=0.8f for TE_{11} mode. Determine the diameter of the waveguide and guide wavelength.

$$f_{c_{TE_{11}}} = \frac{1.84c}{2\pi a}$$

$$0.8f = \frac{1.84c}{2\pi a}$$

2a=3.66cm

$$\lambda_c = \frac{c}{f_c} = \frac{2\pi a}{1.84}$$ =6.25cm

$$\lambda = \frac{c}{f}$$ =5cm

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_c^{\ 2}} + \frac{1}{\lambda_g^{\ 2}}$$

$$\lambda_g = 8.33cm$$

4.Prove that area of circular waveguide is 2.2 times of area of rectangular waveguide for dominant mode propagation

For rectangular waveguide $$f_{cTE_{10}} = \frac{c}{2a}$$

For circular waveguide $$f_{c_{TE_{11}}} = \frac{1.84c}{2\pi r}$$

$$\frac{1.84c}{2\pi r} = \frac{c}{2a}$$

$$a = 1.7r$$

Area of rectangular waveguide $$A_R = a \times b = a \times \frac{a}{2} = \frac{a^2}{2} = \frac{1.7^2 r^2}{2} = 1.445r^2$$

Area of circular waveguide $$A_C = \pi r^2$$

$$\frac{A_C}{A_R} = \frac{\pi r^2}{1.445r^2} = 2.2$$

3 sets

$$Q = \frac{\beta}{2\alpha}$$

Quality factor related to conductor losses is

$$Q_c = \frac{\beta}{2\alpha_c}$$

$$\alpha_c = \frac{R_s}{W Z_0} \quad NP/cm$$

where $R_s = \sqrt{\frac{\omega \mu}{2\sigma}}$, $Z_0 = \frac{h}{W}\sqrt{\frac{\mu}{\epsilon}} = \frac{h}{W}\frac{120\pi}{\sqrt{\epsilon_{re}}}$

$$R_s = \sqrt{\frac{\pi f \mu}{\sigma}} = 2\pi\sqrt{\frac{f_{GHz}}{\sigma}}$$

$$Q_c = \frac{\beta W Z_0}{2R_s} = \frac{\beta W \frac{h}{W}\frac{120\pi}{\sqrt{\epsilon_{re}}}\sqrt{\sigma}}{4\pi\sqrt{f_{GHz}}}$$

$$Q_c = \frac{\beta h 120\pi\sqrt{\sigma}}{4\pi\sqrt{f_{GHz}}\sqrt{\epsilon_{re}}}$$

where $\beta = \frac{2\pi}{\lambda_g} = \frac{2\pi}{\lambda}\sqrt{\epsilon_{re}}$; $\lambda = \frac{c}{f} = \frac{30}{f_{GHz}}$

$$Q_c = \frac{2\pi h 120\pi\sqrt{\sigma}\, f_{GHz}\sqrt{\epsilon_{re}}}{30 \times 4\pi\sqrt{f_{GHz}}\sqrt{\epsilon_{re}}} = 2\pi h\sqrt{\sigma f_{GHz}}$$

$$Q_c = 6.28 h\sqrt{\sigma f_{GHz}}$$

If h is expressed in cm $\boxed{Q_c = 0.628 h\sqrt{\sigma f_{GHz}}}$

Quality factor related to Dielectric loss is

$$Q_d = \frac{\beta}{2\alpha_d}$$

$$\alpha_d = 27.3\frac{\epsilon_r}{\epsilon_{re}}\frac{\tan\theta}{\lambda_g}$$

$$Q_d = \frac{\beta\,\epsilon_{re}\,\lambda_g}{2 \times 27.3\,\epsilon_r \tan\theta}$$

$$Q_d = \frac{2\pi\,\epsilon_{re}\,\lambda_g}{\lambda_g \times 2 \times 27.3\,\epsilon_r \tan\theta} \quad \left[\because \beta = \frac{2\pi}{\lambda_g}\right]$$

$$Q_d = \frac{2\pi\,\epsilon_{re}}{2 \times 27.3\,\epsilon_r \tan\theta}$$

$$Q_d \approx \frac{1}{\tan\theta}$$